Smoothness Properties of Varieties of Borels and Schubert Varieties*

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Abstract

Let G be a connected reductive group defined over an algebraically closed field k, T a fixed maximal torus in G, and B a fixed Borel subgroup containing T, W the Weyl group of G relative to T, and S the set of simple reflections in W defined by (T, B). We denote by X the projective variety of Borel subgroups of G. Let $O(w)$, $w \in W$, be the orbit of $(B, {}^wB) \in X \times X$ under the left action of G. Now we define $\overline{O}(s_1, \ldots, s_n)$, $s_i \in S$, to be the closed subvariety of X^{n+1} whose points are the sequences $(B_0, B_1, \ldots, B_n)$, $B_i \in X$, where $(B_{i-1}, B_i) \in \overline{O(s_i)}$ for $i = 1, \ldots, n$. In this paper, we prove that the canonical projections

$$\pi : \overline{O}(s_1, \ldots, s_i) \to \overline{O}(s_1, \ldots, s_{i-1})$$

are $\mathbf{P^1}$-bundles, which implies that the variety $\overline{O}(s_1, \ldots, s_n)$ is smooth over k.

All varieties or schemes and all morphisms are defined over a fixed algebraically closed field k. For the main part, the only points of a variety under consideration are the points rational over k. The context makes it clear when this is not the case. Let G be a connected reductive algebraic group over k, T a fixed maximal torus, and B a fixed Borel subgroup containing T,

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$W = N_G(T)/T$ the Weyl group of G relative to T, and S the set of simple reflections in W defined by (T, B). We denote by X the projective variety of Borel subgroups of G. Its structure of algebraic variety is defined by the canonical G-equivariant bijection ${}^gB = gBg^{-1} \mapsto gB$ of X onto G/B. Note that if $w \in W$, we can also define ${}^wB = wBw^{-1}$ and wB in the usual way, and they do not depend on the representative $\dot{w}$ of w chosen to define them. Let $O(w)$, $w \in W$, be the orbit of $(B, {}^wB) \in X \times X$ under the left action of G. The set$O(w)$, being locally closed ([Bor], 1.8, Proposition, p.53), defines a subvariety of $X \times X$. Now we define $\overline{O}(s_1, \ldots, s_n)$, $s_i \in S$, to be the closed subvariety (resp. $O(s_1, \ldots, s_n)$, $s_i \in S$, to be the subvariety) of X^{n+1} whose points are the sequences $(B_0, B_1, \ldots, B_n)$, $B_i \in X$, where $(B_{i-1}, B_i) \in \overline{O(s_i)}$ for $i = 1, \ldots, n$ (resp. $(B_{i-1}, B_i) \in O(s_i)$ for $i = 1, \ldots, n$). In particular, for $n = 0$, $\overline{O}() = O() = X$.

Such varieties were first introduced by Deligne and Lusztig as a technical tool in representation theory (See [D-L], 9.1, p.147). Lately they have appeared in other contexts (e.g. [Han], [Han1]). In this paper, we prove that the canonical projections

$$\pi : \overline{O}(s_1, \ldots, s_i) \to \overline{O}(s_1, \ldots, s_{i-1}) \tag{1}$$

are $\mathbf{P^1}$-bundles, which implies that the variety $\overline{O}(s_1, \ldots, s_n)$ is smooth over k. In a special case, this result seems to be implicit in [D-L], 9.2, p.148, but a complete proof has not appeared in print. As a corollary, we obtain desingularizations of $\overline{O(w)}$ and of the Schubert varieties considered by Demazure ([Dem]).

We call a $\mathbf{P^n}$-*bundle* a morphism of schemes $f : E \to Y$ such that for an open covering U_α of Y, $E|U_\alpha = f^{-1}(U_\alpha)$ is U_α-isomorphic to $U_\alpha \times \mathbf{P^n}$, i.e. there is an isomorphism of $E|U_\alpha$ onto $U_\alpha \times \mathbf{P^n}$ compatible with the projections of these two schemes onto U_α. As indicated at the beginning, all schemes and morphisms are defined over k. This notion coincides with that of a locally trivial fiber space with fiber $\mathbf{P^n}$ and structure group $\mathbf{PGL_{n+1}}$, the full group of automorphisms of $\mathbf{P^n}$ ([Gro], ch.IV, 4.7).

Before proceeding, note that the projections π in (1) are G-equivariant with respect to the left action of G.

Lemma 1. *The morphism* $\pi : \overline{O(s)} \to X$*,* $s \in S$*, is a* $\mathbf{P^1}$*-bundle.*

To avoid introducing additional notation, in the course of this proof we will view X as G/B with the pertinent changes for $\overline{O(s)}$ and π. Since π is G-equivariant, it will be enough to find a nonempty open subset V of G/B such that $\overline{O(s)}|V$ is V-isomorphic to $V \times \mathbf{P^1}$. The family $\{gV \mid g \in G\}$ gives the required covering of G/B because each $g \in G$ induces isomorphisms $V \longrightarrow gV$, $\overline{O(s)}|V \longrightarrow g\overline{O(s)}|gV$, and $V \times \mathbf{P^1} \longrightarrow gV \times \mathbf{P^1}$ that allow us to transport the given V-isomorphism $\overline{O(s)}|V \longrightarrow V \times \mathbf{P^1}$ to a gV-isomorphism $g\overline{O(s)}|gV \longrightarrow gV \times \mathbf{P^1}$. Let U be the unipotent part of B, B' the Borel subgroup opposite to B, and U' its unipotent part. We have $U \cap B' = U' \cap B = \{e\}$.

1. The set $U'B$, the big cell, is open in G ([Hum], 28.5, Proposition, p.174); since it is also saturated with respect to the equivalence relation defined by B (i.e. it is a union of B-equivalence classes), the quotient $U'B/B$ is open in G/B. We set $V = U'B/B$ and observe that the canonical bijective quotient morphism

$$\tau : U' \to V = U'B/B$$

is an isomorphism ([Bor], 6.1, Corollary, p.95).

2. Let $P_s = BsB \cup B$ be the minimal parabolic generated by B and s ([Hum], 29.3, Lemma B, p.178). Since $\dim(P_s/B) = 1$, it follows ([Hum], 25.3, Theorem p.154, or [Bor], 13.13, Proposition, p.171) that $P_s/B \cong \mathbf{P^1}$.

3. We need to show that

$$\pi^{-1}(V) = \{(uB, upB) \mid u \in U', p \in P\}.$$

The inclusion of the second set into the first is clear. To prove that the first set is included in the second, pick an element (gB, gsB), $gB = uB$, $g \in G$ and $u \in U'$. Then $g = ub$, and, setting $p = b\dot{s}$ with $\dot{s}$ a representative of s, $bsB = ubsB = upB$.

4. Finally, the map

$$\pi^{-1}(V) \to V \times (P_s/B)$$

$$(uB, upB) \mapsto (uB, pB)$$

is well-defined as a set-theoretic map given that the $u \in U'$ is uniquely determined by uB, and pB is obtained by multiplying upB by u^{-1} on the right. To prove that this map is a morphism, we first notice that

$$\pi^{-1}(V) \subset V \times (G/B).$$

Call pr_1 and pr_2 the projections of $V \times G/B$ onto V and G/B, $\text{inv} : U' \to U'$ the inverse, i.e. $\text{inv}(u) = u^{-1}$, and recall that $\tau : U' \to V$ was the canonical isomorphism. With these notations, our map becomes

$$(\text{pr}_1, (\text{inv} \circ \tau^{-1} \circ \text{pr}_1) \cdot \text{pr}_2)$$

with the domain restricted to $\pi^{-1}(V)$ and the codomain restricted to $V \times (G/B)$ if necessary. This proves that our map is a morphism. We leave to the reader to define the inverse in the obvious way and to show that it is a morphism. Clearly both morphisms are V-morphisms. $\square$

Remark 2. We mention that $\overline{O(s)}$ is also a $\mathbf{P^1}$-bundle with respect to the second projection. This can be seen easily by starting with the new pair of maximal torus ${}^{s^{-1}}T = T$ and Borel ${}^{s^{-1}}B$ instead of T and B.

In the future we want to identify

$$\overline{O}(s_1, \dots, s_i) = \overline{O}(s_1, \dots, s_{i-1}) \times_X \overline{O}(s_i)$$

as schemes, where the morphisms into X defining the right hand side are respectively the last and the first projections. The reader can verify that the canonical map $(B_0, \dots, B_i) \mapsto ((B_0, \dots, B_{i-1}), (B_{i-1}, B_i))$ is an isomorphism on the level of varieties (=reduced schemes), and the only technical point left to check is that $\overline{O}(s_1, \dots, s_{i-1}) \times_X \overline{O}(s_i)$ is reduced. This is true because this fiber product is a $\mathbf{P^1}$-bundle over $\overline{O}(s_1, \dots, s_{i-1})$, being obtained from the $\mathbf{P^1}$-bundle $\overline{O}(s_i) = \overline{O(s_i)}$ by extending the base from X to $\overline{O}(s_1, \dots, s_{i-1})$. By induction, we can also get the identification

$$\overline{O}(s_1, \dots, s_i) = \overline{O}(s_1) \times_X \cdots \times_X \overline{O}(s_i)$$

with the appropriate projections. Now, combining these identifications with 1, we get the following theorem.

Theorem 3. *The morphism*

$$\pi : \overline{O}(s_1, \dots, s_n) \to \overline{O}(s_1, \dots, s_{n-1}),$$

$s_i \in S$, *is a* $\mathbf{P^1}$*-bundle.*

We call the sequence

$$\overline{O}(s_1, \dots, s_n) \to \overline{O}(s_1, \dots, s_{n-1}) \to \cdots \to \overline{O}(s_1) \to \overline{O}() = X$$

the *iterated* $\mathbf{P^1}$*-bundle* over X corresponding to $(s_1, \dots, s_n)$.

Corollary 4. *The variety* $\overline{O}(s_1, \dots, s_n)$, $s_i \in S$, *is irreducible and smooth of dimension* $\dim(X) + n$ *over* k, *and* $O(s_1, \dots, s_n)$ *is dense open in* $\overline{O}(s_1, \dots, s_n)$.

We recall that, if $E \longrightarrow Y$ is a $\mathbf{P^1}$-bundle, and Y is smooth over k (resp. irreducible over k), then E is smooth over k (resp. irreducible over k). For the irreducibility, see, for instance, [EGA], IV, 2.3.5 (iii). By considering the iterated $\mathbf{P^1}$-bundle over X corresponding to $(s_1, \dots, s_n)$, the theorem reduces to the fact that X itself is irreducible and smooth over k, which follows from the connectedness of G and the transitivity of the action of G on X. The assertion about the dimension is clear. The fact that $O(s_1, \dots, s_n)$ is open in $\overline{O}(s_1, \dots, s_n)$ follows easily from the definitions. Since $\overline{O}(s_1, \dots, s_n)$ is irreducible and $O(s_1, \dots, s_n)$ is nonempty and open, it is also dense. □

Now let $w = s_1 \dots s_n$, $s_i \in S$, $i = 1, \dots, n$, be a *reduced decomposition* of $w \in W$. We have a commutative diagram

$$\begin{array}{ccc} \overline{O}(s_1, \dots, s_n) & \xrightarrow{\quad\phi\quad} & \overline{O(w)} \\ & {\scriptstyle\varpi}\searrow \qquad \swarrow{\scriptstyle\pi} & \\ & X & \end{array} \tag{2}$$

where ϕ is the morphism defined by $\phi(B_0, \dots, B_n) = (B_0, B_n)$ (See [D-L], 9.1, p.148 and [D-L], 1.2(a), p.106), and ϖ and π are the first projections. In the following corollary, by *desingularization*, we mean with Grothendieck ([EGA], IV, 7.9.1) a proper birational morphism (consequently surjective) of a nonsingular variety into another, possibly singular, variety.

Corollary 5. *With the notations above, always assuming that* $w = s_1 \dots s_n$ *is a reduced decomposition, we have:*

(i) The morphism $\phi : \overline{O}(s_1, \dots, s_n) \longrightarrow \overline{O(w)}$ *is a desingularization.*

(ii) For any $x \in X$, *the restriction* $\phi_x : \varpi^{-1}(x) \longrightarrow \pi^{-1}(x)$ *of* ϕ *to the fibers of* ϖ *and* π *over* x *is a desingularization.*

(i) This result is known and appears in a more precise form in [D-L], 9.1, p.148. We sketch an argument in the present context. It is clear that the morphism ϕ is proper, being a morphism of projective varieties. Moreover ϕ induces an isomorphism $\phi^0 : O(s_1, \dots, s_n) \xrightarrow{\sim} O(w)$ since $w = s_1 \dots s_n$ is a reduced decomposition ([D-L], p.106). The rest follows from Corollary 4.

(ii) As in (i), the morphism ϕ_x is proper, being a morphism of projective varieties. The commutative diagram (2) restricts to

$$\begin{array}{ccc} O(s_1, \ldots, s_n) & \xrightarrow[\sim]{\phi^0} & O(w) \\ & \varpi^0 \searrow \quad \swarrow \pi^0 & \\ & X & \end{array}$$

In this situation, the restriction $\phi_x^0 : (\varpi^0)^{-1} \longrightarrow (\pi^0)^{-1}$ of ϕ^0 to the fibers over x is also an isomorphism. On the other hand

$$(\varpi^0)^{-1}(x) = \varpi^{-1}(x) \cap O(s_0, \ldots, s_n)$$

is nonempty and open, and consequently dense in $\varpi^{-1}(x)$. Similarly

$$(\pi^0)^{-1}(x) = \pi^{-1}(x) \cap O(w)$$

is nonempty and open, and consequently dense in $\pi^{-1}(x)$. $\square$

Remark 6. For $w \in W$, let $S(w)$ be the image of the Bruhat cell $C(w) = BwB$ under the morphism $G \longrightarrow G/B \longrightarrow X$, $g \mapsto gB \mapsto gBg^{-1}$. The closure $\overline{S(w)}$ is called the *Schubert variety* corresponding to w. Then $S(w) = \{{}^{bw}B \mid b \in B\}$ and, if $x_B = B \in X$,

$$(\pi^0)^{-1}(x_B) = \{({}^{g}B,\, {}^{gw}B) \mid {}^{g}B = B\} = \{B\} \times S(w).$$

Thus we can identify $\overline{S(w)} = \pi^{-1}(x_B)$ and we can regard the morphism ϕ_{x_B} as a desingularization of the Schubert variety $\overline{S(w)}$.

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