

A note on gaps between zeroes of L-functions

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Assuming the Riemann hypothesis, H. Montgomery [M] stated his well-known pair-correlation conjecture

$$\sum_{\substack{0 < \gamma, \gamma' < T \\ \frac{2\pi a}{\log T} \leq \gamma - \gamma' \leq \frac{2\pi b}{\log T}}} 1 \sim \left(\int_a^b \left(1 - \left(\frac{\sin(\pi x)}{\pi x} \right)^2 \right) dx + \delta(a, b) \right) N(T), \quad (1)$$

where

$$\delta(a, b) := \begin{cases} 1, & 0 \in [a, b], \\ 0, & \text{otherwise,} \end{cases}$$

and $N(T)$ counts the zeros (according to multiplicity) $\beta + i\gamma$ with $0 \leq \gamma \leq T$ of the Riemann zeta function $\zeta(s)$. This conjecture is of considerable interest in prime number theory, as has been shown (for example) by D. Goldston [G], P. Gallagher and J. Mueller [GM], and R. Heath-Brown [H-B].

If $L(s)$ is an L-function satisfying axioms I-VI below then one might expect that the following analog of (1) holds:

$$\sum_{\substack{0 < \gamma, \gamma' < T \\ \frac{2\pi a}{a_0(L)\log T} \leq \gamma - \gamma' \leq \frac{2\pi b}{a_0(L)\log T}}} 1 \sim \left(\int_a^b \left(1 - \left(\frac{\sin(\pi x)}{\pi x} \right)^2 \right) dx + \delta(a, b) \right) N_L(T), \quad (2)$$

where $N_L(T)$ is the counting function for the zeros of $L(s)$. It is conceivable that such a statement might eventually be of some relevance to the study of Sato-Tate-type distribution questions of semi-simple conjugacy classes in L-groups associated to the unramified components of an automorphic representation. The main result of this note implies that, for the L-functions satisfying axioms I-VI (e.g., all Dirichlet L-functions), we have a lower bound

$$\lim_{\substack{a \rightarrow 0+ \\ b \rightarrow \infty}} \sum_{\substack{0 < \gamma, \gamma' < T \\ \frac{2\pi a}{a_0(L)\log T} \leq \gamma - \gamma' \leq \frac{2\pi b}{a_0(L)\log T}}} 1 \succeq \left(\int_0^\infty \left(1 - \left(\frac{\sin \pi x}{\pi x} \right)^2 \right) dx + \delta(a, b) \right) N_L(T), \quad (3)$$

where $f(T) \succeq g(T)$ means $\lim_{T \rightarrow \infty} f(T)/g(T) \geq 1$ and the other unexplained notation is defined below. We do not assume a generalized Riemann hypothesis for this.

Let us index the zeros $\beta + i\gamma$ as $\beta_n + i\gamma_n$, ordered lexicographically and counted according to multiplicity. The asymptotic inequality (3) is a consequence of our main result which states that, for L-functions satisfying axioms I-VI below and for sufficiently large but fixed $r > r_0(L)$,

$$\frac{1}{N_L(T)} \#\{0 < \gamma_n < T \mid \frac{\gamma_{n+r} - \gamma_n}{r} \frac{a_0(L) \log \gamma_n}{2\pi} > 1 + c_1 \frac{\sqrt{\log r}}{r}\} > e^{-c_2 r^2}, \quad (4)$$

and

$$\frac{1}{N_L(T)} \#\{0 < \gamma_n < T \mid \frac{\gamma_{n+r} - \gamma_n}{r} \frac{a_0(L) \log \gamma_n}{2\pi} < 1 - c_1 \frac{\sqrt{\log r}}{r}\} > e^{-c_2 r^2}, \quad (5)$$

where c_i are constants depending only on L and $a_0(L)$ is defined below. Again, we do not assume a generalized Riemann hypothesis for this. For the Riemann zeta function, A. Fujii [F] stated such a result with $\frac{\sqrt{\log r}}{r}$ replaced by e^{-cr^2} and $e^{-c_2 r^2}$ replaced by some (undetermined) $\delta_r > 0$. (However, his result holds for all $r \geq 1$.) Again for the Riemann zeta function, A. Selberg proved a slightly stronger result assuming the Riemann hypothesis (unpublished but see [J], Theorem 4.5, chapter 3).

Now let us state the axioms on which our results depend.

- (I) The Dirichlet series $L(s)$ converges absolutely in some half-plane $Re(s) > 1$ and may be meromorphically continued to the entire plane, except possibly for a simple pole at $s = 1$.
- (II) Aside from “trivial” zeros on the negative real axis, all zeros of $L(s)$ lie in the “critical strip” $0 \leq Re(s) \leq 1$, and

$$N_{tr}(T) := \#\{\beta + i\gamma \mid L(\beta + i\gamma) = 0, -T < \beta < 0\} \sim A_L \cdot T, \quad (6)$$

and

$$\begin{aligned} N_L(T) &:= \#\{\rho = \beta + i\gamma \mid L(\beta + i\gamma) = 0, 0 < \gamma < T\} \\ &\sim \frac{a_0(L)}{2\pi} \cdot T \log \frac{T}{2\pi} + \frac{a_1(L)}{2\pi} \cdot T + a_2(L) + S_L(T) + O_L(1/T), \end{aligned} \quad (7)$$

as $T \rightarrow \infty$. Here

$$S_L(T) := \frac{1}{\pi} \arg L(1/2 + it),$$

where \arg is taken in the same way as the Riemann zeta function (as the variation starting from $+\infty$, where the argument is zero, ...).

- (III) There is an Euler product

$$L(s) = \prod_p L_p(s), \quad Re(s) > 1,$$

where $L_p(s)$ is a rational function of p^{-s} , such that

$$\frac{L'}{L}(s) = - \sum_{n \geq 2} a_n \Lambda(n) n^{-s}, \quad \operatorname{Re}(s) > 1,$$

where $a_n \in \mathbb{C}$ are constants satisfying

$$|a_p| < C, \quad \text{for all primes } p,$$

for some $C = C(L)$, and $\Lambda(n)$ is the von Mangoldt function.

(IV) $L(s)$ satisfies the growth conditions:

$$\frac{L'}{L}(s) = \sum_{|s-\rho| \leq 2} \frac{1}{s-\rho} + O_L(\log T), \quad -1 < \operatorname{Re}(s) < 2; \quad (8)$$

for some $\sigma_n \rightarrow \infty$,

$$\frac{L'}{L}(-\sigma_n + it) \ll_L \sigma_n \log T, \quad |t| < T; \quad (9)$$

for all $0 \leq \sigma \leq 1$,

$$L(\sigma + it) \ll_L T^{c(L)}, \quad |t| < T, \quad (10)$$

for some constant $c(L) > 0$ with $T > 1$.

(V) The Rankin convolution of L with itself, $\tilde{L}(s)$, satisfies axioms (I-IV), where

$$\tilde{L}(s) := \sum_n |c_n|^2 n^{-s},$$

if

$$L(s) = \sum_n c_n n^{-s}, \quad \operatorname{Re}(s) > 1.$$

(VI) Let $T^{1/2+\epsilon} \leq H \leq T$, $x := T^{\epsilon/100k}$, and $0 < h = h(T) < 1$. For any positive integer $k \leq (\log \log T)^{1/10}$ we have

$$\begin{aligned} \frac{1}{H} \int_T^{T+H} (S_L(t+h) - S_L(t))^{2k} dt &= \frac{(2k)!}{(2\pi)^{2k} k!} (r(\tilde{L}) \log(3+h \log T))^k \\ &\quad + O_L((Ak)^{2k} (\log(3+h \log T))^{k-1/2}), \end{aligned}$$

where $r(\tilde{L})$ denotes the residue of $\tilde{L}(s)$ at $s = 1$.

1 Preliminary lemma

Our result relies on a distribution theorem for $S_L(t+h) - S_L(t)$, $h = c/\log t$, similar to the one given by A. Ghosh [Gh].

Lemma 1. *Assume $L(s)$ satisfies axioms I-VI above. Let $0 < h = h(T)$ be bounded as $T \rightarrow \infty$ and suppose $h \log T > C_0(L)$ is sufficiently large. Let*

$$M_T = \log \log(h \log T) / \log \log \log(h \log T),$$

and let $b(r) := \pi^{-r-1} \Gamma(\frac{r+1}{2})$. Then for any real number r , $1 \leq r \leq (\log M_T)^{1/4}$, we have

$$\frac{1}{T} \int_0^T \left| \frac{S(t+h) - S(t)}{\sqrt{r(\tilde{L}) \log(h \log T)}} \right|^r dt = b(r)(1 + O_L((\log M_T)^{-1/2})),$$

as $T \rightarrow \infty$.

proof For $h \ll 1$, we let

$$B_h(t) := |S_L(t+h) - S_L(t)| / \sqrt{r(\tilde{L}) \log(h \log T)},$$

and recall that $2^{-r} \Gamma(r+1) / \Gamma(r/2+1) = \pi^{-1/2} \Gamma(r/2+1/2)$. Thus, by axiom VI,

$$\frac{1}{T} \int_0^T B_h(t)^{2k} dt = b(2k) + O((A_L k)^{4k} (\log(3+h \log T))^{-1/2}),$$

for integers k , such that $1 \leq k \leq M_T$.

Choose N be an integer with $N < (\log M_T)^{3/4}$ and write, by the Taylor expansion,

$$(\sin u)^4 = \frac{1}{8} \sum_{j=2}^N b_j u^{2j} + O\left(\frac{(4u)^{2N+2}}{(2N+2)!}\right),$$

where $b_j := \frac{(-4)^{j+1}}{(2j)!} (4^{j-1} - 1)$. As in A. Ghosh [Gh], §2, we obtain for $r \leq \sqrt{\log M_T}$,

$$\begin{aligned} & \frac{1}{T} \int_0^T B_h(t)^r dt - b(r) \\ & \ll_L b(2m) X^{-1-\theta} + (A_L M)^{4M} (\log(3+h \log T))^{-1/2} X^{-1-\theta} e^{X^2} \\ & + \frac{16^N X^{2N+1-\theta}}{N(2N+2)!} [b(2M+2) + (A_L M)^{4M} (\log(3+h \log T))^{-1/2}], \end{aligned}$$

where $r = 2m + 1 + \theta$, $0 < \theta \leq 2$, $M = m + N$, and N is as above. Taking $r \leq (\log M_T)^{3/4}$, $N = (\log M_T)^{3/4}$, and $X = (\log M_T)^{1/4}$, we obtain the estimate desired. \square

Theorem 2. *Let*

$$\mu_T(x, B_h) := \frac{1}{T} \text{meas}\{0 \leq t \leq T \mid B_h(t) \leq x\}$$

and let

$$\mu(x) := \frac{2}{\sqrt{\pi}} \int_0^{\pi x} e^{-t^2} dt.$$

If $0 \ll h \ll 1$ and if $h \log T$ is sufficiently large then

$$|\mu_T(x, B_h) - \mu(x)| \ll (\log \log M_T)^{1/2}.$$

In other words, for each $c > 0$, we have

$$\begin{aligned} & \text{meas}\{0 \leq t \leq T \mid |S(t+h) - S(t)| < \frac{c}{\pi} \sqrt{r(\tilde{L}) \log(3 + h \log T)}\} \\ &= T \left(\frac{2}{\sqrt{\pi}} \int_0^{\pi x} e^{-t^2} dt + O\left(\frac{1}{\sqrt{\log M_T}}\right) \right). \end{aligned}$$

proof Let N denote the integral part of $M_T^{1/8}$ and consider

$$\begin{aligned} \Phi_T(y) &:= \frac{1}{T} \int_0^T e^{iyB_h(t)} dt \\ &= \sum_{j=0}^{N-1} \frac{(iy)^j}{j!} \frac{1}{T} \int_0^T B_h(t)^j dt + O\left(\frac{|y|^N}{N!} \frac{1}{T} \int_0^T B_h(t)^N dt\right) \\ &= \sum_{j=0}^{N-1} \frac{(iy)^j}{j!} b(j) + O\left(\frac{|y|^N}{N!} b(N)\right) + O\left(M_T^{-1/4} \sum_{j=1}^{N-1} \frac{|y|^j}{j!} b(j)\right). \end{aligned}$$

Since

$$b(j) = \pi^{-j-1/2} \Gamma\left(\frac{j+1}{2}\right) = 2\pi^{-j-1/2} \int_0^\infty t^j e^{-t^2} dt,$$

this is (as in [Gh], p. 101)

$$\begin{aligned} & \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \sum_{j=0}^{N-1} \frac{(iyt/\pi)^j}{j!} dt + O\left(M_T^{-1/4} \sum_{j=1}^{N-1} \frac{|y|^j}{j!} b(j)\right) + O\left(\frac{|y|^N}{N!} b(N)\right) \\ & \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} e^{iyt/\pi} dt + O\left(M_T^{-1/4} \sum_{j=1}^{N-1} \frac{|y|^j}{j!} b(j)\right) + O\left(\frac{|y|^N}{N!} b(N)\right), \end{aligned}$$

so

$$\Phi_T(y) - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} e^{iyt/\pi} dt \ll M_T^{-1/4} \sum_{j=1}^{N-1} \frac{|y|^j}{j!} b(j) + \frac{|y|^N}{N!} b(N).$$

As in [Gh], pp. 100-102, the Berry-Essen theorem implies

$$\mu_T(x, B_h) - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} e^{iyt/\pi} dt \ll (\log M_T)^{-1/2},$$

as desired. \square

Now let us prove (4) and (5). Let $h = \frac{2\pi C}{T}$, with C sufficiently large. By the above lemma,

$$\text{meas}\{0 \leq t \leq T \mid |S(t+h) - S(t)| > \frac{1}{20} \sqrt{r(\tilde{L}) \log C}\} > \frac{1}{30} e^{-C^2} T,$$

for $T > T_0(C, L)$. This implies that for a proportion of at least $\frac{1}{3} e^{-C^2}$ of the $t \in (0, T)$, we have

$$\left| N_L\left(t + \frac{2\pi C}{a_o(L) \log T}\right) - N_L(T) - C \right| > \frac{1}{20} \sqrt{r(\tilde{L}) \log C},$$

for C sufficiently large. Since the average of $N_L(t + \frac{2\pi C}{a_0(L)\log T}) - N_L(T)$ is C , we may suppose without loss of generality that our positive proportion in fact satisfies

$$N_L(t + \frac{2\pi C}{a_0(L)\log T}) - N_L(T) > C + \frac{1}{20}\sqrt{r(\tilde{L})\log C}.$$

Let $C = r - \frac{1}{35}\sqrt{\log r}$, so that

$$N_L(t + \frac{2\pi r - \frac{1}{35}\sqrt{\log r}}{a_0(L)\log T}) - N_L(T) > r,$$

for the above proportion of $t \in (0, T)$. This implies (4). This actually implies (5) since the average gap $\gamma_{n+r} - \gamma_n$ is $\frac{2\pi r}{a_0(L)\log \gamma_n}$. \square

Theorem 3. Fix ϵ , $0 < \epsilon < 1/10$ arbitrarily small. For $C > C_0(\epsilon, L)$, $r_0(L) < r < (1 - 3\epsilon)C$ fixed, and $T > T_0(\epsilon, C, L)$, we have

$$\frac{1}{N_L(T)}\#\{0 < \gamma_n < T \mid \gamma_{n+r} - \gamma_n < \frac{2\pi C}{a_0(L)\log \gamma_n}\} > 1 - O\left(\frac{1}{\sqrt{\log \log \log \log C}}\right).$$

proof We have

$$\gamma_{n+r} - \gamma_n < \frac{2\pi C}{a_0(L)\log \gamma_n}$$

for a positive proportion δ_r of γ'_n s if and only if

$$N_L\left(\gamma_n + \frac{2\pi C}{a_0(L)\log \gamma_n}\right) - N_L(\gamma_n) > r$$

for the same positive proportion δ_r of γ'_n s. By axiom (II), we have

$$N_L\left(\gamma_n + \frac{2\pi C}{a_0(L)\log \gamma_n}\right) - N_L(\gamma_n) - C = S_L\left(\gamma_n + \frac{2\pi C}{a_0(L)\log \gamma_n}\right) - S_L(\gamma_n) + O_L\left(\frac{C}{\log \gamma_n}\right),$$

so to prove the theorem we need only show that

$$S_L\left(\gamma_n + \frac{2\pi C}{a_0(L)\log \gamma_n}\right) - S_L(\gamma_n) > r - C + O_L\left(\frac{C}{\log \gamma_n}\right)$$

occurs for a proportion of at least $1 - O\left(\frac{1}{\sqrt{\log \log \log \log C}}\right)$ of the γ'_n s. This follows easily from Theorem 2. \square

Corollary 4. As $C \rightarrow \infty$, we have

$$\lim_{T \rightarrow \infty} \frac{1}{N_L(T)}\#\{0 < \gamma, \gamma' < T \mid 0 < \gamma - \gamma' < \frac{2\pi C}{a_0(L)\log T}\} \geq C.$$

proof By Theorem 3,

$$\begin{aligned}
& \frac{1}{N_L(T)} \#\{0 < \gamma, \gamma' < T \mid 0 < \gamma - \gamma' < \frac{2\pi C}{a_0(L) \log T}\} \\
& \geq \sum_{r=r_0(L)}^{(1-\epsilon)C} \frac{1}{N_L(T)} \#\{0 < \gamma_n < T \mid 0 < \gamma_{n+r} - \gamma_n < \frac{2\pi C}{a_0(L) \log T}\} \\
& \geq (1 - \epsilon - \frac{r_0(L)}{C}) C (1 - O(\frac{1}{\sqrt{\log \log \log \log C}})) \\
& \geq (1 - 2\epsilon) C,
\end{aligned}$$

for sufficiently large C . \square

2 Appendix: Errata to [J]

There is a gap in [J], Ch. 4, as was brought to my attention by D. Hejhal. To prove the results in [J], pp. 117-127, in addition to those axioms listed above, one must assume the following:

Missing Axiom: We have, for $H > T^{1/2+\epsilon}$, and $\sigma \geq 1/2$,

$$N_L(\sigma, T+H) - N_L(\sigma, T) \ll_{L,\epsilon} \left(\frac{\sqrt{H}}{T}\right)^{\frac{1-2\sigma}{4}} H \log H, \quad (11)$$

where

$$N_L(\sigma, T) := \#\{\rho = \beta + i\gamma \mid L(\beta + i\gamma) = 0, 0 < \gamma < T, \beta \geq \sigma\}.$$

First, this “missing axiom” is a well-known result of A. Selberg [S1] in the case $L(s) = \zeta(s)$. (It also seems to be known for Dirichlet L-series, L-functions of quadratic fields and Langlands standard L-functions for $GL(2)$ [S2]. Selberg has other results assuming this axiom and another axiom vaguely related to our axiom (V).)

Second, axiom (VI) above is a well-known result of A. Selberg [S1] in the case $L(s) = \zeta(s)$ and is mentioned in the remark to [J], Proposition 4.2 (on page 158, $S(t+h) - S(t)$ should be $S_L(t+h) - S_L(t)$; in the remark there, $\log k$ should be $\log x$). Thus the results of this paper and [J] are theorems in case $L(s)$ is a Dirichlet L-function. It seems reasonable to expect that they should also hold for Langlands L-functions associated to tempered generic automorphic representations of $GL(n)$.

Lastly, the result (4-5) was listed as Conjecture 4.4 in [J], p. 84. There is a typo there: on page 84, line 4, $C \log r / \sqrt{r}$ should be $\frac{1}{50} \frac{\sqrt{\log r}}{r}$, and on page 84, line 9, $C \log r / r$ should be $C \sqrt{\log r} / r$.

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