Traces of Hecke operators on rational functions

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We compute explicitly the traces of the Hecke operators defined and investigated in Gil and Robins [GR][1]. It turns out, these are not trace class but are the limit of compact integral operators, and their trace can be defined by an appropriate limit. These Hecke operators have trace 1, but behave in some ways analogous to shift operators on ℓ^2.

1 The inner products

Define the **Hardy space** H^2 to be the holomorphic functions f on the open unit disc satisfying

$$||f||_2 = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^2 \, d\theta \right)^{\frac{1}{2}} < \infty.$$

It is well-known (for example, see Rudin [R], ch. 17) that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in H^2 then $||f||_2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty$. The power functions $\{p_n\}_{n=0}^{\infty}$, $p_n(z) = z^n$, form an orthogonal basis.

If f and g are meromorphic in the unit disc, we define

$$(f, g) = \lim_{r \to 1-} \frac{1}{2\pi i} \oint_C f(z) g(z^{-1}) \frac{dz}{z}, \tag{1}$$

and

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[1]These notes were originally written in 2004. They were slight revised in 2008.

$$\langle f, g\rangle = \lim_{r\to 1-} \frac{1}{2\pi i}\oint_C f(z)\overline{g(\overline{z}^{-1})}\frac{dz}{z}, \tag{2}$$

where $C = C_r$ denotes the complex contour over $|z| = r$ tranversed positively $(0 < r < 1)$.

(**Check**: The sup and be swapped with the lim.)

For Laurent series

$$f(z) = \sum_{m\in\mathbb{Z}} a_m z^m, \qquad g(z) = \sum_{n\in\mathbb{Z}} b_n z^n,$$

we have

$$(f, g) = \sum_{m,n} a_m b_n \frac{1}{2\pi i}\oint_C z^m z^{-n-1}\, dz = \sum_n a_n b_n,$$

and

$$\langle f, g\rangle = \sum_{m,n} a_m \overline{b_n} \frac{1}{2\pi i}\oint_C z^m z^{-n-1}\, dz = \sum_n a_n \overline{b_n}.$$

In particular, $\langle ...\rangle$ is a positive-definite inner product.

2 The operators U_p and V_p

Let $p \geq 2$ be a fixed integer (eventually we will assume it is prime) and define

$$U_p : \sum_{m\in\mathbb{Z}} a_m z^m \longmapsto \sum_{m\in\mathbb{Z}} a_{pm} z^m. \tag{3}$$

This is one of the Hecke operators introduced in [GR].

Let $p \geq 1$ be an integer and define

$$\delta_n(p) = \begin{cases} 1, & p|n, \\ 0, & \text{otherwise}, \end{cases}$$

and also

$$V_p : \sum_{m\in\mathbb{Z}} a_m z^m \longmapsto \sum_{m\in\mathbb{Z}} a_m z^{pm} = \sum_{m\in\mathbb{Z}} a_{m/p}\delta_m(p) z^m. \tag{4}$$

This is another Hecke operator introduced in [GR].

Both U_p and V_p are well-defined on H^2. Both fix the constants (which form a subspace of eigenfunctions with eigenvalue 1).

Remark 1 *Other than this eigenspace, these operators behave in some ways like the well-known shift operators on the Hilbert space of square-integrable sequences, ℓ^2. The shift operators have trace* 0 *and have norm* 1*. As we will see, the U_p and V_p would also be traceless if we could somehow ignore the eigenspace of constants. Also, the U_p and V_p have unit norm in the H^2-operator norm.*

Note that V_p can be restricted to the finite dimensional space W_N defined by

$$W_N = \mathbb{C}[z]_N = \{f \in \mathbb{C}[z] \mid \deg f \leq N\}.$$

We define the **trace** of V_p by

$$\mathrm{tr}(V_p) = \lim_{N\to\infty} \mathrm{tr}(V_p|_{W_N}).$$

Let $p_i(z) = z^i$ ($i \geq 0$, $z \in \mathbb{C}$).

Lemma 1 $\mathrm{tr}(V_p) = 1$.

proof: It sufficies to compute the trace of $V_p|_{W_N} : W_N \to W_N$. It is clear that $(V_p p_i, p_i) = 0$, for each $i \geq 1$, whereas $(V_p p_0, p_0) = 1$, so the trace of $V_p|_{W_N}$ is 1. □

Lemma 2 *(a) If f is analytic then*

$$U_p f(w) = (f, K_p^w),$$

where the kernel function K_p^w is given by $K_p^w(z) = (1 - wz^p)^{-1}$.

(b) Likewise,

$$U_p f(w) = \langle f, K_p^w \rangle,$$

where the kernel function K_p^w is given by $K_p^w(z) = (1 - \overline{w}z^p)^{-1}$.

proof: Let $\epsilon_n = 1$ if $n \geq 0$ and $= 0$ otherwise. We have

$$(f, K_p^w) = \sum_{m,n} a_m w^n \epsilon_n \frac{1}{2\pi i} \oint_C z^m z^{-pn-1}\, dz = \sum_{n\geq 0} a_{pn} w^n = U_p f(w).$$

The $\langle ... \rangle$ computation is similar. $\square$

We have the inner product computations

$$(U_p f, g) = \sum_{n\in\mathbb{Z}} a_{pn} b_n = \sum_{n\in\mathbb{Z}} a_n \delta_n(p) b_{n/p} = (f, V_p g),$$

and, likewise, $\langle U_p f, g\rangle = \langle f, V_p g\rangle$. Therefore, V_p is the dual operator of U_p in each of these inner products. Because of this duality, we define the **trace** of U_p by

$$\mathrm{tr}(U_p) = \mathrm{tr}(V_p).$$

Lemma 3 *(a) If f is analytic then*

$$V_p f(w) = (f, K_p^w),$$

where the kernel function K_p^w is given by $K_p^w(z) = (1 - w^p z)^{-1}$. On W_N, one can use instead the truncation of this series, $K_{p,N}^w(z) = \sum_{j=0}^N w^{jp} z^j$.

(b) Likewise,

$$V_p f(w) = \langle f, K_p^w\rangle,$$

where the kernel function K_p^w is given by $K_p^w(z) = (1 - \overline{w}^p z)^{-1}$. On W_N, one can use instead the truncation of this series, $K_{p,N}^w(z) = \sum_{j=0}^{N/p} \overline{w}^{jp} z^j$.

proof: As above,

$$(f, K_p^w) = \sum_{m,n} a_m w^{np} \epsilon_n \frac{1}{2\pi i} \oint_C z^m z^{-n-1}\, dz = \sum_{n\geq 0} a_n w^{pn} = V_p f(w).$$

If $f \in W_N$ then

$$(f, K^w_{p,N}) = \sum_{m\geq 0,\, 0\leq n\leq N/p} a_m w^{np} \frac{1}{2\pi i}\oint_C z^m z^{-n-1}\, dz = \sum_{0\leq n\leq N} \delta_n(p) a_{n/p} w^n = V_p f(w).$$

The rest of the verification is similar to the lemma above, so omitted. □

Since

$$(U_p p_n, p_n) = \langle U_p p_n, p_n\rangle = (V_p p_n, p_n) = \langle V_p p_n, p_n\rangle = \begin{cases} 1, & n = 0, \\ 0, & n > 0, \end{cases}$$

consistent with the results above.

3 The traces

The following is the main result of this note.

Proposition 4

- *The restriction of U_p to the subspace of analytic functions has trace*

$$\mathrm{tr}\, U_p = \frac{1}{2\pi i}\oint_C k_U(z,z)\frac{dz}{z} = 1,$$

where $k_U(z,w) = K^w_p(z)$ is the kernel function given by Lemma 2 above.

- *The restriction of V_p to the subspace of analytic functions has trace*

$$\mathrm{tr}\, V_p = \frac{1}{2\pi i}\oint_C k_V(z,z)\frac{dz}{z} = 1,$$

where $k_V(z,w) = K^w_p(z)$ is given by Lemma 3 above.

The problem is that these operators U_p and V_p are not trace class (see §5.4 in [GGK], for example). Two possible approaches come to mind.

One is to restrict the operators to W_N, polynomials of degree $\leq N$ in H^2, then take limits as the degree $N \to \infty$. Here,

$$\mathrm{tr}\,(V_p|_{W_N}) = \frac{1}{2\pi i}\oint_C K^z_{p,N}(z)\frac{dz}{z} = \frac{1}{2\pi i}\oint_C \sum_{j=0}^{N} z^{jp+j}\frac{dz}{z} = 1,$$

since $V_p|_{W_N}$ is a compact integral operator. Unless there is a mistake, this implies

$$\mathrm{tr}\,(V_p) = \lim_{N\to\infty} \mathrm{tr}\,(V_p|_{W_N}) = 1,$$

which is not what the other computations show.

Another is to replace U_p by $U_{p,r} : \sum_{m\in\mathbb{Z}} a_m z^m \longmapsto \sum_{m\in\mathbb{Z}} r^m a_{pm} z^m$, for a fixed $r \in (0,1)$ (and similarly for V_p). There is an analog of the above integral kernel computation for this new operator and, hopefully, when one takes $r \to 1-$, the limit can be taken inside the integral.

We shall take this second approach.

Lemma 5 *The operator $U_p : H^2 \to H^2$ has trace*

$$\mathrm{tr}\, U_p = \lim_{r\to 1-} \frac{1}{2\pi i} \oint_{C_r} k_U(z,z) \frac{dz}{z} = 1,$$

where C_r is the circle of radius r.

proof: Let $C = C_1$. First, we compute $\frac{1}{2\pi i} \oint_C k_U(z,z) \frac{dz}{z}$. Using the kernel from Lemma 2(a), we have

$$\begin{aligned}
\tfrac{1}{2\pi i} \oint_{C_r} k_U(z,z) \tfrac{dz}{z} &= \tfrac{1}{2\pi i} \oint_{C_r} (1 - z^{pm+1})^{-1} \tfrac{dz}{z} \\
&= \tfrac{1}{2\pi} \int_0^{2\pi} (1 - r^{p+1} e^{i\theta(p+1)})^{-1}\, d\theta \\
&= \tfrac{1}{2\pi i} \oint_{C_1} (1 - r^{p+1} z^{p+1})^{-1} \tfrac{dz}{z} \\
&= \textstyle\sum_i \mathrm{Res}_{z=z_i} \frac{1}{z(1-r^{p+1}z^{p+1})} + \tfrac{1}{2\pi i} \oint_{C_R} (1 - r^{p+1} z^{p+1})^{-1} \tfrac{dz}{z},
\end{aligned}$$

where $R > 1$ is choosen sufficiently large and z_i represent all the poles of the integrand inside the circle of radius R. Taking $R \to \infty$, this contour integral over C_R goes to zero. For any $(p+1)$-st root of unity ζ, we have

$$1 - r^{p+1} z^{p+1} = (1 - rz\zeta^{-1}) \sum_{j=0}^{p} r^j z^j \zeta^{-j} = -r(z - r^{-1}\zeta) \sum_{j=0}^{p} r^j z^j \zeta^{-j},$$

so

$$\sum_\zeta \mathrm{Res}_{z=\zeta/r} \frac{1}{z(1 - r^{p+1} z^{p+1})} = \sum_\zeta \frac{r}{\zeta} \mathrm{Res}_{z=\zeta/r} \frac{1}{1 - r^{p+1} z^{p+1}} = \sum_\zeta \frac{r}{\zeta}(-r)(p+1) = 0.$$

Thus, the above total expression is

$$= \sum_i \mathrm{Res}_{z=z_i} \frac{1}{z(1-r^{p+1}z^{p+1})} = 1-(p+1)\sum_i \zeta = 1,$$

as desired.

Using the kernel from Lemma 2(b), we have

$$\begin{aligned} \tfrac{1}{2\pi i}\oint_C k_U(z,z)\tfrac{dz}{z} &= \tfrac{1}{2\pi i}\oint_C (1-\overline{z}z^p)^{-1}\tfrac{dz}{z} \\ &= \tfrac{1}{2\pi}\int_0^{2\pi}(1-r^{p+1}e^{i\theta(p-1)})^{-1}\,d\theta \\ &= \tfrac{1}{2\pi i}\oint_{C_1}(1-r^{p+1}z^{p-1})^{-1}\tfrac{dz}{z}. \end{aligned}$$

The rest of the analysis is similar, and, again, has limit 1 as $r \to 1-$. □

Similar computations work for V_p (but since V_p is dual to U_p they should have the same trace), and they are omitted.

Note that we have

$$k_U(w,z) = \overline{k_V(z,w)},$$

which is consistent with the fact that $U_p = V_p^*$.

4 The Hecke operator

If $\alpha_p \in \mathbb{C}$ is a constant and we define

$$T_p = U_p + \alpha_p V_p,$$

by analogy with the "usual" Hecke operator, then

$$T_p f(z) = \sum_{n=0}^{\infty}(a_{np} + \epsilon_n(p)\alpha_p a_{n/p})z^n.$$

Since T_p is the sum of traceless operators, it is also traceless.

What about it's spectrum? Since $||U_p|| = ||V_p|| = 1$, in the H^2-operator norm, we must have that $||T_p|| \le 1 + |\alpha_p|$. Therefore, the spectrum of T_p must be contained in the closed disc in $\mathbb{C}$ of radius $1 + |\alpha_p|$ centered at 0.

The eigenvector condition $T_p f(z) = \lambda f(z)$ is equivalent to

$$a_{np} + \epsilon_n(p)\alpha_p a_{n/p} = \lambda a_n. \tag{5}$$

The eigenspaces of those functions in H^2 which have the special form

$$f(z) = \sum_{k=0}^{\infty} a_k z^{p^k} \tag{6}$$

are easy to compute. In this case, the recursive relation (5) takes the form (2)

$$a_{k+1} + \alpha_p a_{k-1} = \lambda a_k. \tag{7}$$

Lemma 6 *If* $|\lambda| \neq 2\sqrt{\alpha_p}$ *then*

$$a_k = a_0 \frac{\lambda^k}{2^{k+1}\beta}[(1+\beta)^{k+1} - (1-\beta)^{k+1}],$$

where $\beta = \sqrt{1 - 4\alpha_p\lambda^{-2}}$. *The associated series (6) is an eigenfunction in* H^2 *if* $|\lambda \pm \sqrt{\lambda^2 - 4\alpha_p}| < 2$.

Remark 2 *Observe that* $|\lambda \pm \sqrt{\lambda^2 - 4\alpha_p}| < 2$ *forces* $|\alpha_p| < 1$ *and (therefore)* $||T_p|| < 2$.

proof: We may assume without loss of generality that $a_0 = 1$. If $a_k = c_1 r_1^k + c_2 r_2^k$, for some constants r_1, r_2, c_1, and c_2 then the recursive relation (7) implies r_i is a root of $r^2 - \lambda r + \alpha_p = 0$ and that

$$c_1 = \frac{1}{2} + \frac{1}{2}\beta^{-1}, \quad c_2 = \frac{1}{2} - \frac{1}{2}\beta^{-1}.$$

Using the quadratic formula to compute r_i gives the result claimed. □

In general, the sequence $\{a_k\}$ is completely determined by a_0, but convergence problems may arise if we want an element of H^2. However, in the vector space of formal power series, such convergence problems don't arise and it is easy to see that the λ-eigenspace of T_p is one-dimensional, provided $|\lambda| \neq 2\sqrt{\alpha_p}$.

References

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