

Principal Ideals and Associate Rings

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Abstract

A commutative ring A with 1 is **associate** provided whenever two elements a and b generate the same principal ideal there is a unit u such that $ua = b$. The main results proved here are:

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- Every commutative Noetherian ring with 1 is a subdirect product of rings which have the property that all their unital subrings are associate.
- Every commutative ring embeds into an associate ring.
- Every commutative unital algebraically closed or principal ideal ring is associate.
- The direct product of associate rings is associate.

One novel feature here is that we prove these results using model theory. For comparison and completeness, the main result (Theorem 14) is also sketched using a localization argument.

1 Ring Theoretic Preliminaries

A **ring** here shall always have an associative multiplication together with a multiplicative identity $1 \neq 0$. Subrings are required to contain the identity 1 of the over ring in question and ring homomorphisms are required to respect unities. When we refer to the direct product of rings, we shall always mean the unrestricted direct product, especially in the case where I is infinite. A subring A of $\bigotimes_{i \in I} A_i$ is a **subdirect product** provided for each fixed $j \in I$, projection onto the j -th coordinate remains an epimorphism with image A_j when restricted to A .

If \mathcal{X} is a nonempty class of rings closed under isomorphism, then we call the ring A **residually- \mathcal{X}** provided to every nonzero element $a \neq 0$ in A there is a ring $B_a \in \mathcal{X}$ and an epimorphism $\phi_a : A \rightarrow B_a$ which does not annihilate a (see [C], page 101). An immediate consequence of the treatment in McCoy (see [Mc], Theorem 3.6, page 52) is

Lemma 1 *Let \mathcal{X} be a nonempty class of rings closed under isomorphism. The ring A is residually- \mathcal{X} if and only if A is embeddable as a subdirect product of an indexed family $\{A_i\}_{i \in I}$ of rings such that $A_i \in \mathcal{X}$ for all $i \in I$.*

It is easy to see that

Lemma 2 *Let \mathcal{X} be a nonempty class of rings closed under isomorphism. The ring A is residually- \mathcal{X} if and only if $\bigcap K = \{0\}$, where the intersection is taken over all two-sided ideals K such that $A/K \in \mathcal{X}$.*

A ring A is called the **direct union** of a family $\mathcal{F} = \{A_\alpha\}_{\alpha \in \Lambda}$ of subrings of A if and only if \mathcal{F} satisfies the following two properties:

- (i) For each $a \in A$ there is at least one $\alpha \in \Lambda$ such that $a \in A_\alpha$, and
- (ii) Whenever $\alpha, \beta \in \Lambda$ there is at least one $\gamma \in \Lambda$ such that $A_\alpha \subseteq A_\gamma$ and $A_\beta \subseteq A_\gamma$.

We call A **finitely generated** provided that, as an associative \mathbb{Z} -algebra with 1, it has at least one finite set of generators. It is easy to see that every ring is the direct union of its finitely generated subrings (see [G], Lemma 3, page 130). Furthermore, a finitely generated commutative ring is a homomorphic image of a polynomial ring $\mathbb{Z}[X_1, \dots, X_n]$ (see [L], page 60) and so, as an immediate consequence of the Hilbert Basis Theorem, must be Noetherian. It follows that every commutative ring is the direct union of a family of Noetherian subrings.

Finally a commutative ring A is **algebraically closed** provided given any commutative extension $E \supseteq A$ it is the case that any finite system

$$p_i(X_1, \dots, X_n) = 0, \quad i = 1, \dots, k \quad (p_i \in A[X_1, \dots, X_n], \quad 1 \leq i \leq k)$$

of equations which has a solution $(X_1, \dots, X_n) = (e_1, \dots, e_n) \in E^n$ must already have a solution $(X_1, \dots, X_n) = (a_1, \dots, a_n) \in A^n$ (see [H], page 47). It follows from Theorem 3.2.1 of Hodges (see [H], page 48), that every commutative ring can be embedded in an algebraically closed commutative ring.

2 Model Theoretic Preliminaries

Let L_{Ring} be the first-order language whose only relation symbol is $=$, always to be interpreted as the identity relation, and whose only function and constant symbols are: two binary operation symbols $+$ and \cdot , a unary operations symbol $-$, and two constant symbols 0 and 1. (For a more complete description of such a language see [BS] or [CK].) Observe that the axioms for a commutative ring can be formalized within L_{Ring} . Recall that a **sentence** of L_{Ring} is a formula of L_{Ring} containing no free occurrences of any variable.

We define two increasing chains of sets of formulas of L_{Ring} ,

$$\Sigma_0 \subseteq \Sigma_1 \text{ and } \Pi_0 \subseteq \Pi_1 \subseteq \Pi_2$$

as follows.

A formula is in Π_0 if and only if it is logically equivalent to a quantifier free formula of L_{Ring} . A formula is in Σ_0 if and only if it is in Π_0 . A formula is in Π_1 if either it is in Π_0 or is logically equivalent to a formula of the form $\forall \bar{x}\phi(\bar{x})$, where \bar{x} is a tuple of variables and $\phi(\bar{x})$ is in Σ_0 . A formula is in Σ_1 if either it is in Σ_0 or is logically equivalent to a formula of the form $\exists \bar{x}\phi(\bar{x})$, where \bar{x} is a tuple of variables and $\phi(\bar{x})$ is in Π_0 . A formula is in Π_2 if either it is in Π_1 or is logically equivalent to a formula of the form $\forall \bar{x}\phi(\bar{x})$, where \bar{x} is a tuple of variables and $\phi(\bar{x})$ is in Σ_1 . A formula which is simultaneously a Π_n -formula and a sentence of L_{Ring} is a Π_n -**sentence** of L_{Ring} . Similarly, a formula which is simultaneously a Σ_n -formula and a sentence of L_{Ring} is a Σ_n -**sentence** of L_{Ring} .

A **universal sentence** is one of the form $\forall \bar{x}\phi(\bar{x})$, where \bar{x} is a tuple of variables and $\phi(\bar{x})$ is a quantifier free formula of L_{Ring} . An **existential sentence** is one of the form $\exists \bar{x}\phi(\bar{x})$ with similar provisos. A **universal-existential sentence** of L_{Ring} is one of the form $\forall \bar{x}\exists \bar{y}\phi(\bar{x}, \bar{y})$ with similar provisos.

Allowing vacuous quantifications and regarding each of $\forall x\phi$ and $\exists x\phi$ as logically equivalent to ϕ if the variable x does not occur in the formula ϕ , we may assert that a sentence of L_{Ring} is Π_1 if and only if it is logically equivalent to a universal sentence of L_{Ring} , a sentence of L_{Ring} is Σ_1 if and only if it is logically equivalent to an existential sentence of L_{Ring} , and that a sentence of L_{Ring} is Π_2 if and only if it is logically equivalent to a universal-existential sentence of L_{Ring} .

A set T of sentences of a language is called a **theory** in the language. Henceforth we tacitly assume all sets of sentences of L_{Ring} under consideration contain the ring axioms. A **model** of a theory T of L_{ring} is a ring satisfying all the sentences of T (see [H], page 9). Let T be a theory of L_{Ring} . T is said to be **inconsistent** if at least one **contradiction**, i.e., a formula of the form $\psi \wedge \sim \psi$, is a formal consequence of T ; otherwise, T is said to be **consistent**. The Gödel-Henkin Completeness Theorem asserts that T is consistent if and only if T has a model (see [BS], page 102, or [CK]). In the event that T is a consistent theory in L_{Ring} containing the ring axioms, we may form the **model class** $\mathbb{M}(T)$ consisting of all rings satisfying all of the sentences T and be assured that $\mathbb{M}(T)$ is not empty. It will be convenient for us to summarize here some of the preservation theorems of classical model theory. (See [CK] and [G]).

Lemma 3 *Let T be a consistent set of sentences of L_{Ring} . Then*

1. $\mathbb{M}(T)$ is closed under the formation of subrings if and only if there is a consistent set S of Π_1 -sentences of L_{Ring} such that $\mathbb{M}(T) = \mathbb{M}(S)$.
2. $\mathbb{M}(T)$ is closed under the formation of extensions if and only if there is a consistent set S of Σ_1 -sentences of L_{Ring} such that $\mathbb{M}(T) = \mathbb{M}(S)$.¹
3. $\mathbb{M}(T)$ is closed under the formation of direct unions if and only if there is a consistent set S of Π_2 -sentences of L_{Ring} such that $\mathbb{M}(T) = \mathbb{M}(S)$.

Lemma 3 may be paraphrased by asserting that a model class is closed under

1. Subrings if and only if it has a set of universal axioms;
2. Extensions if and only if it has a set of existential axioms;
3. Direct unions if and only if it has a set of universal-existential axioms.

Recall that $\Pi_1 \subseteq \Pi_2$ (or that universal sentences are considered degenerate cases of universal-existential sentences). We see that every class axiomatized by a consistent set of Π_1 -sentences (or universal sentences, if one prefers) of L_{Ring} is closed under the formation of direct unions.

Given a consistent theory T in L_{Ring} we let T_{\forall} be the set of all Π_1 -sentences of L_{Ring} which are formal consequences of T . T_{\forall} is surely consistent since every model A of T must also be a model of T_{\forall} . Furthermore, $\mathbb{M}(T_{\forall})$ is closed under the formation of subrings since it has a set of Π_1 -axioms. Thus, every subring of a model of T is a model of T_{\forall} . The converse is Corollary 3.1.2. of Hodges (see [H], page 37), which we here record as a lemma.

Lemma 4 : *If T is a consistent set of sentences of L_{Ring} then $\mathbb{M}(T_{\forall})$ consists of precisely the subrings of models of T .*

¹This is to be interpreted in the present context as asserting that other than the ring axioms the remaining sentences in S (if any) are all in Σ_1 . Moreover by extensions, of course, we mean ring extensions. (One direction of this requires Corollary 4, page 273, of [G].)

3 Associate and Superassociate Rings

Definition 5 Let A be a commutative ring with identity. A is **associate** provided whenever $a, b \in A$ and $Aa = Ab$ there is a unit $u \in A$ such that $ua = b$. The associate ring A is **superassociate** provided every subring of A is associate.

Remark 1 (A) We will give an example of an associate ring which is not superassociate.

(B) Every integral domain is superassociate.

(C) Exercises requesting the construction of nonassociate rings have appeared earlier in the literature as have explicit constructions of nonassociate rings. (See [A] (Exercise 7, page 106), [B] (Example 5.15, page 42), [F] (page 580) and [K] (counter-examples (a), (b), page 466).)

Theorem 6 A principal ideal ring with identity is associate.

The theorem is proven in [B], page 194. For completeness, we provide a brief sketch.

By Theorem 33 of [ZS], a principal ideal ring is a direct product of principal ideal domains and “special principal ideal rings” (see [ZS], page 245). It can be shown without too much trouble using a result on page 245 of [ZS] that a special principal ideal ring is an associate ring. We conclude from this and Theorem 7 that a principal ideal ring with 1 is associate.

The following facts are easy to prove directly.

Theorem 7 Let $A = \bigotimes_{i \in I} A_i$ (unrestricted direct product). Then A is associate if and only if A_i is associate for all $i \in I$.

Theorem 8 The class of associate rings is closed under the formation of direct limits.

(Here by a direct limit we can take the direct union of a family of rings - see p. 130 of [G]. However, the theorem is true more generally even in cases where the direct limit is not the direct union.)

Lemma 9 The class of associate rings is not closed under the formation of subrings, subdirect products, or homomorphic images.

Remark 2 *The erroneous assertion that all commutative rings are associate appears in the literature (see [CF], page 118).*

Proof: Consider the rings $R_0 = \mathbb{Z}[X, Y]$ and $R = \mathbb{Z}[X, Y]/(X - XY^2)$. Clearly, R_0 is associate. We will show its homomorphic image R is not associate. We will also embed R as a subdirect product in the associate ring $\mathbb{Z}[Y] \otimes \mathbb{Z}[X] \otimes \mathbb{Z}[X]$.

Letting x and y be the images of the indeterminates X and Y respectively under the canonic epimorphism $R_0 \rightarrow R$, it is not difficult to show that every element of R is uniquely of the form $xp(x) + q(y) + xyr(x)$ where p, q and r are integral polynomials in a single indeterminate.

If R were associate then, since $xR = xyR$, there would exist a unit $u \in R$ satisfying $xy = ux$. This implies that there exist polynomials $F, G, U, V \in R_0$ such that

$$\begin{aligned} U(X, Y)V(X, Y) &= 1 + (X - XY^2)F(X, Y), \\ U(X, Y)X &= XY + (X - XY^2)G(X, Y), \end{aligned}$$

where U maps to u under the map $R_0 \rightarrow R$ and V maps to u^{-1} . These imply $U(X, Y) - Y = (1 - Y^2)G(X, Y)$. Let $\bar{U}(Y)$ denote the image of $U(X, Y)$ under the canonical map $R_0 \rightarrow R_0/XR_0 \cong \mathbb{Z}[Y]$. Similarly for $\bar{V}(Y), \bar{G}(Y)$. The above equations imply

$$\bar{U}(Y)\bar{V}(Y) = 1, \quad \bar{U}(Y) - Y = (1 - Y^2)\bar{G}(Y).$$

The first equation implies $\bar{U}(Y) = \pm 1$. If $\bar{U}(Y) = 1$ then the second equation implies $1 = (1 + Y)\bar{G}(Y)$, a contradiction. If $\bar{U}(Y) = -1$ then the second equation implies $-1 = (1 - Y)\bar{G}(Y)$, a contradiction. These contradictions show R is not associate.

There are canonical epimorphisms

$$\begin{aligned} \alpha : R_0 &\rightarrow R_0/XR_0 \cong \mathbb{Z}[Y] \\ \beta : R_0 &\rightarrow R_0/(1 - Y)R_0 \cong \mathbb{Z}[X] \\ \gamma : R_0 &\rightarrow R_0/(1 + Y)R_0 \cong \mathbb{Z}[X]. \end{aligned}$$

Letting $f = \alpha \otimes \beta \otimes \gamma$ we get a homomorphism

$$f : R_0 \rightarrow \mathbb{Z}[Y] \otimes \mathbb{Z}[X] \otimes \mathbb{Z}[X].$$

If $a \in R_0$ lies in the kernel of f then a is a multiple of X , of $1 - Y$, and of $1 + Y$. Since R_0 is a unique factorization domain ([ZS], vol I, page 32), it follows that a is a multiple of $X - XY^2$. This implies that f induces an injection

$$\phi : R \rightarrow \mathbb{Z}[Y] \otimes \mathbb{Z}[X] \otimes \mathbb{Z}[X].$$

Thus R embeds into the direct product $\mathbb{Z}[Y] \otimes \mathbb{Z}[X] \otimes \mathbb{Z}[X]$ of integral domains. In fact ϕ embeds R as a subdirect product. To see this, notice that α, β and γ are onto mappings, so that the composite mappings of ϕ and the projections are surjective.

Thus, although direct products of integral domains are associate, they need not be superassociate. \square

Definition 10 *A commutative ring A will be called **domainlike** provided every zero divisor is nilpotent.*

Note that every integral domain is domainlike.

Lemma 11 *Every domainlike ring is superassociate.*

Proof: Assume A is domainlike. Let B be a subring of A and assume $a, b \in B$ generate the same principal ideal in B . If $Ba = 0 = Bb$, then $a = b = 0$ and $1 \cdot a = b$. Assume $Ba = Bb$ is a nonzero ideal in B . Then $a \neq 0$ and $b \neq 0$. Now there are elements $u, v \in B$ such that $ua = b$ (since $b \in Ba$) and $vb = a$ (since $a \in Bb$). Then $uvb = b$ and $(uv - 1)b = 0$. But $b \neq 0$ so $uv - 1$ is a zero divisor. Since A is domainlike $r = uv - 1$ is nilpotent so there is an integer $n > 0$ such that $r^n = 0$. But then it is well-known that $1 + r$ is a unit ($(1 + r)^{-1} = 1 - r + \dots + (-r)^{n-1}$) and so, since $uv = 1 + r$, each of u and v individually is a unit as well. \square

Remark 3 *There are superassociate rings which are not domainlike. For example, let R be a unital Boolean ring with more than 2 elements. A partial order on R is well-defined by $a \leq b$ provided $ab = a$. Thus, $a \in (b)$ implies $a \leq b$ and $b \in (a)$ implies $b \leq a$, whence $a = b$. Thus R is superassociate. If $a \neq 0, 1$ then $a(1 - a) = 0$, so a is a zero divisor which is not nilpotent. Thus R is not domainlike.*

Theorem 12 *Every Noetherian commutative ring with identity is residually superassociate.*

Proof: From the Noether-Lasker Primary Decomposition Theorem (see [L], Theorem 4, page 154), it follows that in any Noetherian commutative ring A , (0) has a primary decomposition, i.e., the trivial ideal is a *finite* intersection of primary ideals,

$$(0) = \bigcap_{i=1}^r Q_i.$$

Q_i being primary (see [L] p.152) means that A/Q_i is domainlike and hence is superassociate by Lemma 11. This is sufficient, according to Lemma 2, for A to be residually superassociate. \square

We next observe that the class of associate rings is the model class $\mathbb{M}(T)$ of a set T of sentences of L_{Ring} . T may be taken to be the commutative ring axioms together with the single additional axiom

$$\forall x_1 \forall x_2 \forall y_1 \forall y_2 \exists z_1 \exists z_2 \\ ((y_1 x_1 = x_2) \wedge (y_2 x_2 = x_1)) \rightarrow ((z_1 x_1 = x_2) \wedge (z_2 x_2 = x_1) \wedge (z_1 z_2 = 1)).$$

In particular, we prove

Lemma 13 *Any universal sentence (Π_1 -sentence) of L_{Ring} true in every associate ring is true in any commutative ring. In other words, every commutative ring is a model of T_\forall .*

Proof: Let ϕ be any universal sentence of L_{Ring} true in every associate ring. A commutative ring A is the direct union of its finitely generated subrings. Since universal sentences are preserved by direct unions (see Lemma 3 (3) and subsequent discussion), it will suffice to show that ϕ is true in every finitely generated subring of A in order to deduce that ϕ holds in A . But a finitely generated subring A_0 of A is Noetherian. By Theorem 7 and Lemma 1, such a ring A_0 is embeddable into an unrestricted direct product B of an indexed family $\{B_i\}_{i \in I}$ of associate rings. By Theorem 7, B is associate. Thus, ϕ is true in B . Since universal sentences hold in subrings whenever they hold in an extension, ϕ must be true in A_0 . Since A_0 was an arbitrary finitely generated subring of A and since A is the direct union of such, ϕ must be true in A . Since A was an arbitrary commutative ring and since ϕ was an arbitrary universal sentence of L_{Ring} true in every associate ring we must have that every commutative ring is a model of T_\forall . \square

By Lemmas 4 and 13, we have therefore proven

Theorem 14 *Every commutative ring is embeddable in an associate ring.*

Corollary 15 *Every algebraically closed commutative ring with 1 is associate.*

Proof: Let A be an algebraically closed commutative ring. By Theorem 14 there is an associate ring B into which A embeds. Suppose $a, b \in A$ are such that $Aa = Ab$. Then there are elements $c, d \in A$ such that $ca = b$ and $db = a$. But then $Ba = Bb$ and since B is associate the system of equations

$$\begin{cases} xa = b \\ yb = a \\ xy = 1 \end{cases} \quad (1)$$

has a solution in B . Since A is algebraically closed in the class of commutative rings, (1) must have a solution in A . Thus, there is a unit u of A such that $ua = b$. Hence, A is associate. \square

4 A Purely Algebraic Proof and an Open Question

This section sketches a second proof of our main result Theorem 14. It uses standard ring-theoretical arguments involving localization, rather than the model-theoretic proof above using the interplay between logic and ring theory. We thank an anonymous referee for this sketch.

We make four observations from which Theorem 14 follows. Let R denote a unital commutative ring.

1. In a commutative ring, any ideal embeds in a maximal ideal.
2. Localization of R at a maximal ideal M (or any prime ideal) gives a local ring R_M .
3. If M is a maximal ideal containing the annihilator of a , then a does not map to zero under the canonical map $R \rightarrow R_M$.

This follows from Proposition 2.1 in [E].

4. If R is a local ring then R is associate.

Let R be a local ring, M the maximal ideal of R , and let $a, b \in R$ satisfy $aR = bR$. If $a = bx$ and $b = ay$, for some $x, y \in R$, then $a(1 - xy) = 0$. If $1 - xy \notin M$ then it must be invertible, so $a = b = 0$. If $1 - xy \in M$ then x and y must be units.

For each nonzero $a \in R$, let M_a denote a maximal ideal of R containing the annihilator of a and let R_a denote the localization of R at M_a . As was observed above, the image of a under the canonical map $R \rightarrow R_a$ is nonzero. This implies that the map

$$R \hookrightarrow \bigotimes_{a \in R - \{0\}} R_a \quad (2)$$

is an injection.

Now each R_a is associate. Thus by Theorem 7, $\bigotimes_{a \in R - \{0\}} R_a$ is associate. So from (2) we conclude our result, Theorem 14.

We now end our paper with the following question which to the best of our knowledge remains open:

Question: If A is an associate ring and X is an indeterminate over A , is the polynomial ring $A[X]$ still an associate ring?

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