Representations of finite groups on the Riemann-Roch space of a toric variety

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Abstract

We study the action of a finite group on the Riemann-Roch space of a toric variety. Our main result is the following: if G is a finite subgroup of the (toric) automorphism group of a complete, smooth toric variety X and if D is a divisor on X left stable by G then we can write the natural representation of G on $L(D)$ is a direct sum of explicitly described irreducible subrepresentations. We also determine explicitly the trace of this representation.

Contents

Let X be a smooth projective variety over an algebraically closed field F and let G be a finite subgroup of automorphisms of X over F. We often identify X with its set of F-rational points $X(F)$. If D is a divisor [1] of X which G leaves fixed then G acts on the Riemann-Roch space $L(D)$. We ask: which (modular) representations arise in this way? If π is such a representation, what is its trace?

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[1] Since X is smooth, we need not distinguish between Weil divisors and Cartier divisors (Hartshorne [H], Prop. II.6.11, and Remark II.6.11.1A).

In the case when G acts on a curve X, this question has been investigated by many mathematicians - Hurwitz, Chevalley and Weil, among others. (We refer, for example, to [JT] for further references.) In this short note, we discuss the simpler case of a group acting (as toric automorphisms) on a toric variety.

1 The action of G on $L(D)$

Let X be a smooth projective variety over an algebraically closed field F. Let $F(X)$ denote the function field of X (the field of rational functions on X) and, if D is any divisor on X, let

$$L(D) = \{f \in F(X)^\times \mid (f) + D \geq 0\} \cup \{0\},$$

and let $\ell(D)$ denote its dimension. This definition makes sense even if X is merely normal and D is a Weil divisor.

The action of $\mathrm{Aut}(X)$ on $F(X)$ is defined by

$$\begin{array}{rccc} \rho: & \mathrm{Aut}(X) & \to & \mathrm{Aut}(F(X)), \\ & g & \longmapsto & (f \longmapsto f^g) \end{array}$$

where $f^g(x) = (\rho(g)(f))(x) = f(g^{-1}(x))$.

Of course, $\mathrm{Aut}(X)$ also acts on the group $Div(X)$ of divisors of X, denoted $g : D \longmapsto g(D) = \sum_P d_P g(P)$, for $g \in \mathrm{Aut}(X)$ and $D = \sum_P d_P P \in Div(X)$. It is easy to show that $(f^g) = g(f)$, where (f) denotes the (principal) divisor of the function $f \in F(X)$. Because of this, if $(f) + D \geq 0$ then $(f^g) + g(D) \geq 0$, for all $g \in \mathrm{Aut}(X)$.

If $G \subset \mathrm{Aut}(X)$ leaves $D \in Div(X)$ fixed then we denote the induced representation of G in $L(D)$ again by ρ:

$$\rho : G \to \mathrm{Aut}(L(D)).$$

2 An example

Let Δ be a fan in a lattice L. Denote by τ_1, ..., τ_n the edges or rays of the fan and let v_i denote the first (smallest) lattice point along the ray τ_i. Let D_i denote the Weil divisor

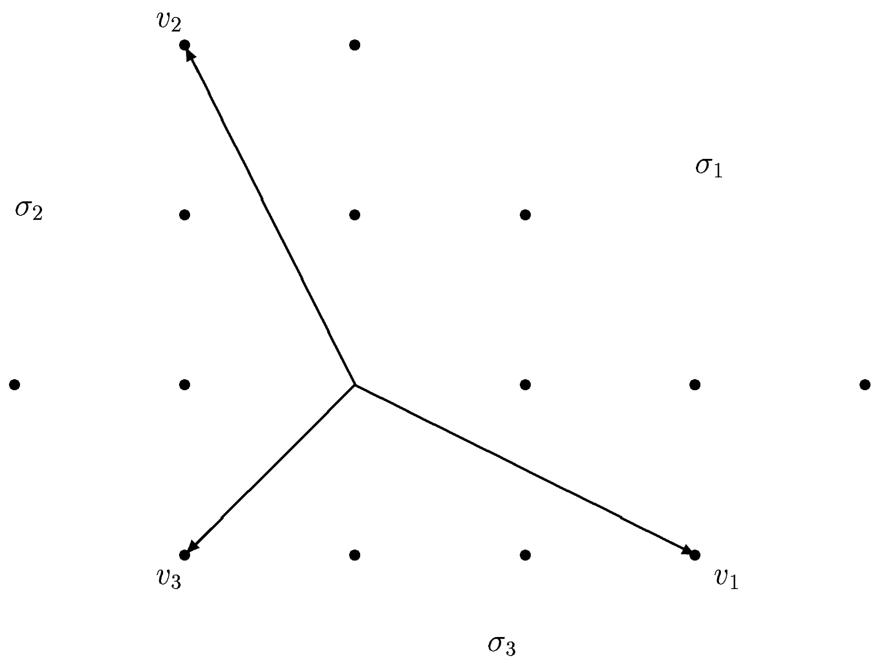
$$D_i = \mathrm{Hom}(\tau_i^\perp \cap L^\perp, \mathbb{C}^\times),$$

which maybe regarded as the closure of the orbit of T acting on the edge τ_i.

Let Δ be the fan generated by

$$v_1 = 2e_1 - e_2, \quad v_2 = -e_1 + 2e_2, \quad v_3 = -e_1 - e_2.$$

In the notation above, the divisor $D = d_1D_1 + d_2D_2 + d_3D_3$ is a Cartier divisor [2] if and only if $d_1 \equiv d_2 \equiv d_3 \pmod 3$.



Let G denote the automorphism group of Δ generated by the map g sending (x, y) to (y, x), swapping v_1 and v_2 and leaving v_3 fixed. By Theorem 1.13 in Oda [O], this corresponds to a T-equivariant automorphism of $X(\Delta)$.

Let

$$\begin{aligned} P_D &= \{(x, y) \mid \langle (x, y), v_i \rangle \geq -d_i, \ \forall i\} \\ &= \{(x, y) \mid 2x - y \geq -d_1, -x + 2y \geq -d_2, -x - y \geq -d_3\} \end{aligned}$$

denote the polytope associated to the Weil divisor $D = d_1D_1 + d_2D_2 + d_3D_3$, where D_i is as above.

[2]This is an Exercise on page 65 of [F], the solution of which is an easy calculation using the Exercise on page 62, which is in turn, basically solved in the back of the book.

Let $d_1 = d_2 = 6$ and $d_3 = 0$. Then P_D is a triangle in the plane with vertices at $(-6,-6)$, $(-2,2)$, and $(2,-2)$. Note that it remains invariant under the action of G. Moreover, the G action on $P_D \cap L^\perp$ has 7 singleton orbits (the lattice points $(-i,-i)$, where $0 \leq i \leq 6$) and 12 orbits of size 2.

In this example, the patch U_{σ_1} is an affine variety with coordinates x_1, x_2, x_3 given by $x^3 - x_2x_3 = 0$. The automorphism g acts on U_{σ_1} sending (x_1, x_2, x_3) to (x_1, x_3, x_2). The torus embedding $T \hookrightarrow U_{\sigma_1}$ is given by sending (t_1, t_2) to $(x_1, x_2, x_3) = (t_1t_2, t_1t_2^2, t_1^2t_2)$.

The patch U_{σ_2} is an affine variety with coordinates y_1, y_2, y_3 given by $y^2 - y_1y_3 = 0$. The automorphism g does not act on U_{σ_2}. The torus embedding $T \hookrightarrow U_{\sigma_2}$ is given by sending (t_1, t_2) to $(y_1, y_2, y_3) = (t_1^{-2}t_2^{-1}, t_1^{-1}, t_1^{-1}t_2)$.

The patch U_{σ_3} is an affine variety with coordinates z_1, z_2, z_3 given by $z^2 - z_1z_3 = 0$. The automorphism g does not act on U_{σ_3}. The torus embedding $T \hookrightarrow U_{\sigma_3}$ is given by sending (t_1, t_2) to $(x_1, x_2, x_3) = (t_1^{-1}t_2^{-2}, t_2^{-1}, t_1t_2^{-1})$. The automorphism g sends U_{σ_2} to U_{σ_3} by sending (y_1, y_2, y_3) to (z_1, z_2, z_3).

3 The main result

The following is our main result. Let Δ be a fan associated to an integral lattice L in $\mathbb{R}^n$. Let P_D denote the polytope associated to the Weil divisor D on $X = X(\Delta)$.

Theorem 1 *Let $X = X(\Delta)$ be a complete toric variety defined over an algebraically closed field F. Suppose $G \subset \mathrm{Aut}(X)$ is a finite subgroup, either* $\mathrm{char}(F) = 0$ *or $p = \mathrm{char}(F)$ does not divide $|G|$, and that the Weil divisor D on X is fixed by G. Let P_D denote the polytope associated to the Weil divisor D on X and let $S = P_D \cap L^\perp$. Let*

$$S = S_1 \cup S_2 \cup \ldots \cup S_k$$

denote the decomposition of S into disjoint primitive G-orbits. For each $m \geq 1$, the natural representation π of G on $L(D)$ decomposes into a direct sum of k irreducible permutation representations

$$\pi = \oplus_{i=1}^k \pi_i,$$

where π_i is $|S_i|$-dimensional.

proof: First, note that there is a natural isomorphism $L(D) \cong \Gamma(X, \mathcal{O}(D))$ (see for example Griffiths and Harris [GH], page 136). We use this map to pull back the action of G on $\Gamma(X, \mathcal{O}(D))$ to an action on $L(D)$. By Fulton [F], page 66, we have

$$L(D) = \oplus_{u \in S} F \cdot \chi^u,$$

where $\chi = (x_1, ..., x_n)$ and χ^u is the associated monomial in multi-index notation. This shows that π, with representation space $V = L(D)$ has an $|S_i|$-dimensional subrepresentation π_i, with representation space $V_i = \oplus_{u \in S_i} F \cdot \chi^u$. If F has characteristic p and p does not divide $|G|$ then every finite dimensional representation of G is semi-simple (Mascke's Theorem, Thrm 3.14, [CR], or [Se], §15.7). From these, the theorem follows. □

Question: Given G, X, and D as above. Do the (equivalence classes of) representations of G occurring in $L(D)$ determine D up to linear equivalence?

Let $(\ldots, \ldots)$ denote the inner product on $V = L(D) = \oplus_{u \in S} F \cdot \chi^u$ given by

$$(\sum_{u \in S} a_u \chi^u, \sum_{v \in S} b_v \chi^v) = \sum_{w \in S} a_w b_w.$$

What is the trace of the representation π in the above theorem?

Theorem 2 *Let X, G, D, S and F be as in the above theorem. Assume, in addition, F has characteristic* 0. *Then*

$$\mathrm{tr}(\pi(g)) = |\{u \in S \mid g(u) = u\}| = |Fix_S(g)|.$$

proof: This follows from well-known facts on permutation representations. □

For example, the trace of the representation π in §2 is

$$\mathrm{tr}(\pi(g)) = \begin{cases} 31, & g = 1 \\ 7, & g \neq 1. \end{cases}$$

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