Splitting fields of representations of generalized symmetric groups, with examples

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Contents

1. Introduction

The main result of this paper, which is mostly (if not entirely) expository, is to "determine" (in a sense made precise later) the splitting field of any irreducible character of a generalized symmetric group. This was basically solved by M. Benard [B]. We use one of his results to make the splitting field "explicit". These notes formed a basis for some lectures in a special topics course held Fall 1997-98 at the USNA.

Let C_ℓ denote the cyclic group of order $\ell \geq 1$, let S_n denote the symmetric group of degree $n \geq 1$, and let G denote the semi-direct product $G = C_\ell^n \; >\!\!\lhd S_n$. We think of this as the set of pairs (v, p), with

- $v = (v_1, ..., v_n)$, where each $v_i \in C_\ell = \{0, 1, ..., \ell - 1\}$,
- $p \in S_n$,
- S_n acts on C_ℓ^n by $p(v) = (v_{p(1)}, v_{p(2)}, ..., v_{p(n)})$,
- multiplication given by

$$(v, p) * (v', p') = (v + p(v'), pp'), \qquad (v, p), (v', p') \in G.$$

Definition 1.1. *A group of the form $C_\ell^n \; >\!\!\lhd S_n$, also written S_n* wr *C_ℓ (here "wr" denotes the wreath product [R]), will be called a* **generalized symmetric group**.

We may identify each element of $C_\ell^n \; >\!\!\lhd S_n$ with an $n \times n$ monomial matrix (a matrix with exactly one non-zero entry in each row and column) with entries in C_ℓ.

Generalized symmetric groups occur in various parts of mathematics - for example, in classifying linear codes up to isometry in the Hamming metric [F] and in the mathematics of the Rubik's cube [J], to mention a few.

Our main goal will be to investigate the following question: What is the "smallest" extension of $\mathbb{Q}$ required to realize (using matrices) a given irreducible character of a generalized symmetric group?

In §3.1 we answer the analogous question over the reals, which turns out to be a little easier. We show that the $m_{\mathbb{R}}(\chi) = 1$, for every irreducible character of a generalized symmetric group[1]. This implies that there are no "quaternionic" representations in this case.

For any finite group G, let G^* denote the set of equivalence classes of irreducible representations of G. If G is abelian then G^* is a group under ordinary multiplication. In case $G = C_\ell^n \; >\!\!\lhd S_n$, the group C_ℓ^* acts on the set G^* by the tensor product. This action determines an equivalence relation on G^*.

[1] Independently, A. Turull communicated to me a different proof than the argument given here.

We begin §3.2 below by collecting, for the sake of comparison, the results on this problem which may be proven as corollaries to special cases of known results. To determine a "nice" explicit matrix representation, for practical purposes it suffices to find one in each C_ℓ^*-equivalence class of G^*. We shall do this by showing, in §3, that for at least one (explicitly given) element of each C_ℓ^*-equivalence class of G^*,

(a) $m_{\mathbb{Q}}(\chi) = 1$, and

(b) determining the field $\mathbb{Q}(\chi)$,

under "fairly general conditions" (given below). We also give an example of a character (which still has Schur index 1 over $\mathbb{Q}$) *not* meeting these "fairly general conditions".

In the last section, §3.3, Benard's result that $m_{\mathbb{Q}}(\chi) = 1$ is used with some other results to determine the field extension of $\mathbb{Q}$ over which an irreducible χ may be realized.

Theorem 1.2. *Let $\chi = tr(\theta_{\rho,\mu})$ be an irreducible character of $G = S_n\ wr\ C_\ell$ as in §2.1. We have*

$$Gal(\mathbb{Q}(\zeta_\ell)/\mathbb{Q}(\chi)) = Stab_\Gamma(\chi).$$

Unexplained notation and definitions shall be given below.

It is known that if F is a field of characteristic $p > 0$ then $m_F(\chi) = 1$ (see Theorem 9.21(b) in [I]). Consequently, a similar result may hold over a field of prime characteristic as well.

2. Background

2.1. Representations of some semi-direct products

The representations of a semi-direct product of a group H by an abelian group A, $G = A >\!\!\lhd H$ (so A is normal in G) can be described explicitly in terms of the representations of A and H. The purpose of this section is to explain how this is done. We follow the presentation of [Se], chapter 8.

Let H be a subgroup of G. Let $f \in R(H)$ be a class function on H. Extend f to G trivially as follows:

$$f^0(g) = \begin{cases} f(g), & g \in H, \\ 0, & g \notin H, \end{cases}$$

for all $g \in G$. This is *not* a class function on G in general. To remedy this, we "average over G" using conjugation: Define the function $f^G = Ind_H^G(f)$ induced by f to be

$$Ind_H^G(f)(g) = \frac{1}{|H|} \sum_{x \in G} f^0(x^{-1}gx) = \sum_{x \in G/H} f^0(x^{-1}gx). \qquad (2.1)$$

Since A is normal in G, G acts on the vector space of formal complex linear combinations of elements of A^*,

$$V = \mathbb{C}[A^*] = span\{\mu \mid \mu \in A^*\},$$

by

$$(g\mu)(a) = \mu(g^{-1}ag), \quad \forall g \in G,\ a \in A,\ \mu \in A^*.$$

We may restrict this action to H, giving us a homomorphism $\phi^* : H \to S_{A^*}$, where S_{A^*} denotes the symmetric group of all permutations of the set A^*. This restricted action is an equivalence relation on A^* which we refer to below as the ***H*-equivalence relation**. Let $[A^*]$ denote the set of equivalence classes of this equivalence relation. If μ, μ' belong to the same equivalence class then we write $\mu' \sim \mu$ (or $\mu' \sim_H \mu$ if there is any possible ambiguity).

Suppose that H acts on A by means of the automorphism given by the homomorphism $\phi : H \to S_A$, where S_A denotes the symmetric group of all permutations of the set A. In this case, two characters $\tau, \tau' \in A^*$ are equivalent if there is an element $h \in H$ such that, for all $a \in A$, we have $\tau'(a) = \tau(\phi(h)(a))$.

When there is no harm, we identify each element of $[A^*]$ with a character of A.

For each $\mu \in [A^*]$, let

$$H_\mu = \{h \in H \mid h\mu = \mu\}.$$

This group is called the stabilizer of μ in H. Let $G_\mu = A > \triangleleft H_\mu$, for each $\mu \in [A^*]$. There is a natural projection map $p_\mu : G_\mu \to H_\mu$ given by $ah \longmapsto h$, i.e., by $p_\mu(ah) = a$.

Extend each character $\mu \in [A^*]$ from H_μ to G_μ trivially by defining

$$\mu(ah) = \mu(a),$$

for all $a \in A$ and $h \in H_\mu$. This defines a character $\mu \in G_\mu^*$. For each $\rho \in H_\mu^*$, say $\rho : H_\mu \to Aut(V)$, let $\tilde{\rho} \in G_\mu^*$ denote the representation of G_μ obtained by "pulling back" ρ via the projection $p_\mu : G_\mu \to H_\mu$, i.e., define $\tilde{\rho} = \rho \circ p_\mu$.

For each $\mu \in [A^*]$ and $\rho \in H_\mu^*$ as above, let

$$\theta_{\mu,\rho} = Ind_{G_\mu}^G (\mu \cdot \tilde{\rho}). \tag{2.2}$$

Finally, we can completely describe all the irreducible representations of G.

Theorem 2.1. *(Proposition 25 in [Se], chapter 8)*

(a) For each $\mu \in [A^*]$ and $\rho \in H_\mu^*$ as above, $\theta_{\mu,\rho}$ is an irreducible representation of G.

(b) Suppose $\mu_1, \mu_2 \in [A^*]$, $\rho_1 \in H_{\mu_1}^*$, $\rho_2 \in H_{\mu_2}^*$. If $\theta_{\mu_1,\rho_1} \cong \theta_{\mu_2,\rho_2}$ then $\mu_1 \sim \mu_2$ and $\rho_1 \cong \rho_2$.

(c) If $\pi \in G^*$ then $\pi \cong \theta_{\mu,\rho}$, for some $\mu \in [A^*]$ and $\rho \in H_\mu^*$ as above.

Definition 2.2. *Let $A = C_\ell^n$. Let $\eta_k(z) = z^k$, for $z \in C_\ell$ and $1 \leq k \leq \ell - 1$. For $\eta \in C_\ell^*$, let $\mu \otimes \eta = (\mu_1\eta, \mu_2\eta, ..., \mu_n\eta)$ where $\mu = (\mu_1, \mu_2, ..., \mu_n)$. This defines an action of C_ℓ^* on A^* and hence on G^*. We call two representations $\theta_{\mu,\rho}$, $\theta_{\mu',\rho'}$ C_ℓ^*-**equivalent**, and write $\theta_{\mu,\rho} \sim_\ell \theta_{\mu',\rho'}$, if $\rho = \rho'$ and $\mu' = \mu \otimes \eta$ for some $\eta \in C_\ell^*$. Similarly, we call two characters μ, μ' of C_ℓ^n C_ℓ^*-**equivalent**, and write $\mu \sim_\ell \mu'$, if $\mu' = \mu \otimes \eta$ for some $\eta \in C_\ell^*$.*

Example 2.3. *Let $\ell = 9$, $n = 3$ and $\mu = (\eta_2, \eta_5, \eta_8)$. Then $\mu \sim \mu \otimes \eta_3$.*

Remark 1. *Let $A = C_\ell^n$ and $H = S_n$. Note that*

$$\theta_{\mu\otimes\eta,\rho} = \theta_{\mu,\rho} \otimes \eta,$$

for $\eta \in C_\ell^*$. Therefore, the matrix representations of two C_ℓ^*-equivalent representations differ only by a character.

2.2. Characters of generalized symmetric groups

Let $G = C_\ell^n >\!\!\lhd S_n$.

The results in the above section tells us how to construct all the irreducible representations of G. We must

1. write down all the characters (i.e., 1-dimensional representations) of $A = C_\ell^n$,

2. describe the action of S_n on A^*,

3. for each $\mu \in [A^*]$, compute the stabilizer $(S_n)_\mu$,

4. describe all irreducible representations of each $(S_n)_\mu$,

5. write down the formula for the character of $\theta_{\mu,\rho}$.

Definition 2.4. *Write $\mu \in [A^*]$ as $\mu = (\mu_1, ..., \mu_n)$, where each component is a character of the cyclic group C_ℓ, $\mu_j \in C_\ell^*$. Let $\mu'_1, ..., \mu'_r$ denote all the distinct characters which occur "in μ", so*

$$\{\mu'_1, ..., \mu'_r\} = \{\mu_1, ..., \mu_n\}.$$

Let n_1 denote the number of μ'_1's "in μ", n_2 denote the number of μ'_2's "in μ", ..., n_r denote the number of μ'_r's "in μ". Then $n = n_1 + ... + n_r$. Call this the partition associated to μ.

Lemma 2.5. *Write $\mu \in [A^*]$ as $\mu = (\mu_1, ..., \mu_n)$, where each component is a character of the cyclic group C_ℓ, $\mu_j \in C_\ell^*$. We have*

$$(S_n)_\mu = S_{n_1} \times ... \times S_{n_r},$$

where $n = n_1 + ... + n_r$ is the partition associated to μ.

Remark 2. *It is known that every character of S_n is not only real-valued but in fact integer-valued (see [Se], chapter 13).*

Combining the above lemma and the fact that every irreducible representation of S_m is real, one can show that every irreducible representation of a direct product of symmetric groups is also real.

If two characters $\mu = (\mu_1, ..., \mu_n)$, $\mu' = (\mu'_1, ..., \mu'_n)$ belong to the same class in $[(C_\ell^n)^*]$, under the S_n-equivalence relation, then their associated partitions are equal. Therefore, we have proven the following

Lemma 2.6. *There is a function $c : [(C_\ell^n)^*] \to \Pi(n)$ such that $c(\mu)$ is the partition of n associated to any character belonging to μ.*

In other words, the partition associated to a character μ really only depends on the S_n-equivalence class of μ in $[(C_\ell^n)^*]$. Moreover, if $\mu \sim_\ell \mu'$, that is they belong to the same C_ℓ^*-equivalence class, then $c(\mu) = c(\mu')$.

Example 2.7. *Let* $G = C_3^4 \;>\!\!\lhd S_4$*. Let* $\mu \in (C_3^4)^*$*,* $\rho \in ((S_4)_\mu)^*$*, and let* $\pi = Ind_H^G(\tilde{\rho}\mu)$ *as above. Then*

$(S_4)_\mu$	*equivalence classes of* $\mu \in (C_3^4)^*$	*dim's of* $\rho \in ((S_4)_\mu)^*$	*number of* π*'s*
S_4	$(\omega, \omega, \omega, \omega)$*,... (3 elements)*	*1,1,2,3,3*	*3+3+3+3+3*
S_3	$(1, \omega, \omega, \omega)$*,...(4 elements)*	*1,1,2*	*6+6+6*
$S_2 \times S_2$	$(\omega, \omega, 1, 1)$*,... (3 elements)*	*1,1,1,1*	*3+3+3+3*
S_2	$(1, 1, \omega, \omega^2)$*,...(3 elements)*	*1,1*	*3+3*

We have

$$\sum_{\pi \in G^*} \chi_\pi(1)^2 = 12^2{\cdot}6+6^2{\cdot}12+4^2{\cdot}6+4^2{\cdot}6+8^2{\cdot}6+1^2{\cdot}3+1^2{\cdot}3+2^2{\cdot}3+3^2{\cdot}3+3^2{\cdot}3 = 81{\cdot}24 = |G|.$$

The Frobenius formula for the character of an induced representation gives

Lemma 2.8. *Let* $G = C_\ell^n \;>\!\!\lhd S_n$ *and let* χ *denote the character of* $\theta_{\mu,\rho}$*. Then*

$$\chi(\vec{v}, p) = \sum_{g \in S_n/(S_n)_\mu} \chi_\rho^o(gpg^{-1})\mu^g(\vec{v}),$$

for all $\vec{v} \in C_\ell^n$ *and* $p \in S_n$*. In particular, if* $p = 1$ *then*

$$\chi(\vec{v}, 1) = (\dim\, \rho) \sum_{g \in S_n/(S_n)_\mu} \mu^g(\vec{v}).$$

Remark 3.

- *Let* $\eta_k(z) = z^k$*, for* $z \in C_\ell$ *and* $1 \leq k \leq \ell - 1$*. For* $\eta \in C_\ell^*$*, let* $\mu \otimes \eta = (\mu_1\eta, \mu_2\eta, ..., \mu_n\eta)$*, as above, where* $\mu = (\mu_1, \mu_2, ..., \mu_n)$*. Then*

$$(v_1 v_2 ... v_n)^k \mathrm{tr}(\theta_{\mu,\rho}(\vec{v}, p) = \mathrm{tr}(\theta_{\mu\otimes\eta_k,\rho}(\vec{v}, p), \tag{2.3}$$

where $\vec{v} = (v_1, ..., v_n)$ *and* $v_i \in C_\ell = \{z \in \mathbb{C} \mid z^n = 1\}$*.*

- *It is clear from the first part of this remark and the previous lemma that if* $\mu \otimes \eta_k \sim_\ell \mu$ *for some* k *then* $\mathrm{tr}(\theta_{\mu,\rho}(\vec{v}, p) = 0$*, for all* $\vec{v} \neq \vec{0}$*.*

2.3. The Frobenius-Schur indicator

Let G be any finite group. In general, there are three "types" of representations of a finite group:

Definition 2.9. *Let $\rho : H \to Aut(W)$ be an n-dimensional irreducible representation of a finite group H on a complex vector space W. Let χ denote the character of ρ.*

Exactly one of the following possibilities must hold:

1. *One of the values of the character χ is not real. Such representations will be called complex (or type 1 or unitary).*

2. *All the values of χ are real and ρ is realizable by a representation over a real vector space. Such representations will be called real (or type 2 or orthogonal).*

3. *All the values of χ are real but ρ is not realizable by a representation over a real vector space. Such representations will be called quaternionic (or type 3 or symplectic).*

Proposition 2.10. *([Se], Proposition 39 in §13.2) Let $\rho : H \to Aut(W)$ be an irreducible representation of a finite group G on a complex vector space W with character χ. Then*

$$\frac{1}{|G|}\sum_{g\in G}\chi(g^2) = \begin{cases} 0, & \rho \text{ complex}, \\ 1, & \rho \text{ real}, \\ -1, & \rho \text{ quaternionic}. \end{cases}$$

(This quantity is sometimes called the Frobenius-Schur indicator of ρ.)

It can be shown (using the above proposition) that if ρ is type i and $\rho \cong \rho'$ then ρ' is also type i.

Lemma 2.11. *Let π denote an irreducible representation of G. If $\pi \cong \pi^*$ then π is either real or quaternionic.*

Lemma 2.12. *$\theta_{\rho,\mu}$ is not complex if and only if μ is self-dual.*

proof: This lemma is proven by calculating the Frobenius-Schur indicator of a representation of a generalized symmetric group.

Let $G = S_n$ wr $C_\ell = C_\ell^n >\!\!\lhd S_n$. Note that if $x = (g, \zeta), y = (h, \zeta') \in G$ then

$$x^2 = (g^2, \zeta + g(\zeta)), \qquad y^{-1} = (h^{-1}, -h^{-1}(\zeta')).$$

The trace of $\theta_{\rho,\mu}(x^2)$ is $\frac{1}{|G_\mu|}$ times

$$\begin{array}{c} \sum_{y\in G} \chi_\rho^o(y^{-1}x^2y)\mu^o(y^{-1}x^2y) \\ = \sum_{h\in S_n} \sum_{\zeta'\in C_\ell^n} \chi_\rho^o(h^{-1}g^2h)\mu^o(h(\zeta) + hg(\zeta) + \zeta' - g^2h(\zeta'))) \\ = \sum_{h\in S_n} \chi_\rho^o(h^{-1}g^2h)\mu(h(\zeta) + hg(\zeta)) \sum_{\zeta'\in C_\ell^n} \mu(\zeta' - g^2h(\zeta'))). \end{array} \tag{2.4}$$

By orthogonality, the last sum is zero unless $\mu = \mu^{g^2h}$, where μ^{g^2h} is the character one obtains by composing μ with the permutation $g^2h : C_\ell^n \to C_\ell^n$.

If $g = 1$ then the inner sum in (2.4) above is only non-zero in case $h \in (S_n)_\mu$. If, in addition, $h \in (S_n)_\mu$ then the inner sum in the last term of (2.4) is equal to $\ell^n = |C_\ell^n|$.

Now let us sum (2.4) over all $x \in G$. First, note that

$$\sum_{\zeta\in C_\ell^n} \mu(h(\zeta) + hg(\zeta)) = 0$$

unless $\mu^g = \mu^{-1}$. If $\mu^g = \mu^{-1}$ and $\mu = \mu^{g^2h}$ then $\mu^h = \mu$.

Therefore, either the sum of (2.4) is zero or there is an element $g_0 \in S_n$ such that $\mu^{g_0} = \mu^{-1} = \overline{\mu}$ (such a character is called self-dual). In particular, $\theta_{\rho,\mu}$ is not complex if and only if μ is self-dual.

2.4. Cyclotomic fields and Tchebysheff polynomials

Though it seems certain this material is known, I know of no reference.

Let n denote a positive integer divisible by 4, let $r = \cos(2\pi/n)$, $s = \sin(2\pi/n)$, and let $d = n/4$. If

$$T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1, \quad ...,$$

denote the Tchebysheff polynomials, defined by $\cos(n\theta) = T_n(\cos(\theta))$, then

$$T_d(r) = 0.$$

Let $\zeta_n = exp(2\pi i/n)$ and let $F_n = \mathbb{Q}(\zeta_n)$ denote the cyclotomic field of degree $\phi(n)$ over $\mathbb{Q}$. If $\sigma_j \in Gal(F_n/\mathbb{Q})$ is defined by $\sigma_j(\zeta_n) = \zeta_n^j$ then

$$Gal(F_n/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times,$$

where $\sigma_j \longmapsto j$.

Lemma 2.13. *Assume n is divisible by 4.*

- *$\mathbb{Q}(r)$ is the maximal real subfield of F_n, Galois over $\mathbb{Q}$ with*

  $$Gal(F_n/\mathbb{Q}(r)) = \{1, \tau\},$$

  where τ denotes complex conjugation. Under the canonical isomorphism

  $$Gal(F_n/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times,$$

  we have

  $$Gal(\mathbb{Q}(r)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\}.$$

- *If n is divisible by 8 then r and s are conjugate roots of T_d. In particular, $s \in \mathbb{Q}(r)$ and $T_d(s) = 0$.*

- *We have $\sigma_j(r) = T_j(r)$*

- *If $n \geq 4$ is a power of 2 then T_d is the minimal polynomial of $\mathbb{Q}(r)$. Furthermore, in this case*

$$\cos(\pi/4) = \sqrt{2}/2,\ \cos(\pi/8) = \sqrt{2+\sqrt{2}}/2,\ \cos(\pi/16) = \sqrt{2+\sqrt{2+\sqrt{2}}}/2,\ \ldots$$

The proof of this lemma is left to the reader.

3. The Schur index

It is a result of Benard [B] that the Schur index over $\mathbb{Q}$ of any irreducible character of a generalized symmetric group is equal to 1. This section recalls, for the sake of comparison with the literature, other results known about the Schur index in this case.

Suppose that G is a finite group and $\pi \in G^*$, $\pi : G \to Aut(V)$, for some complex vector space V. We say that π may be **realized** over a subfield $F \subset \mathbb{C}$ if there is an F-vector space V_0 and an action of G on V_0 such that V and $\mathbb{C} \otimes V_0$ are equivalent representations of G, where G acts on $\mathbb{C} \otimes V_0$ by "extending scalars" in V_0 from F to $\mathbb{C}$. Such a representation is called an **F-representation**. In other words, π is an F-representation provided it is equivalent to a representation which can be written down explicitly using matrices with entries in F.

Suppose that the character χ of π has the property that

$$\chi(g) \in F, \qquad \forall g \in G,$$

for some subfield $F \subset \mathbb{C}$ independent of g. It is unfortunely true that, in general, π is not necessarily an F-representation. However, what is remarkable is that, for some $m \geq 1$, there are m representations, $\pi_1, ..., \pi_m$, all *equivalent to* π, such that $\pi_1 \oplus ... \oplus \pi_m$ is an F-representation. The precise theorem is the following remarkable fact.

Theorem 3.1. *(Schur, [K], chapter 15, or [I], chapter 10) Let χ be an irreducible character and let F be any field containing the values of χ. There is an integer $m \geq 1$ such that $m\chi$ is the character of an F-representation.*

Definition 3.2. *The smallest $m \geq 1$ in the above theorem is called the* **Schur index** *and denoted $m_F(\chi)$.*

The following sections shall list some properties of the Schur index in the case where G is a generalized symmetric group and F is either the reals or rationals.

3.1. No quaternionic representations

Theorem 3.3. *Let $G = S_n$ wr $C_\ell = C_\ell^n >\!\lhd S_n$. There are no quaternionic representations in G^*.*

sketch: This result follows from Theorem 15.1.1 and Proposition 14.1.8 of [Ka], which we recall below for the reader's convenience. To be more precise, we must introduce some notation:

- let $\mathbb{R}(\pi) = \mathbb{R}(\chi)$ denote the extension field of $\mathbb{R}$ obtained by adjoining all the values of $\chi(g)$ ($g \in G$), where χ is the character of π,

- let $\nu(\pi) = \nu(\chi)$ denote the Frobenius-Schur indicator of π (so $\nu(\pi) = \frac{1}{|G|}\sum_{g\in G}\chi(g^2))$,
- let $m_{\mathbb{R}}(\pi) = m_{\mathbb{R}}(\chi)$ denote the Schur multiplier of π (by definition, the smallest integer $m \geq 1$ such that $m\pi$ can be realized over $\mathbb{R}$ (this integer exists, by the above-mentioned theorem of Schur).

Proposition 3.4. *(***Proposition 14.1.8***, [K], page 809) Let χ be an irreducible character of G and let ψ denote an irreducible character of a subgroup H of G. If $<\psi^G, \chi>= 1$ then $m_{\mathbb{R}}(\chi)$ divides $m_{\mathbb{R}}(\psi)$.*

Theorem 3.5. *(***Theorem 15.1.1***, [K], page 839) Let χ be an irreducible character of G. Then $m_{\mathbb{R}}(\chi) = 1$, if $\nu(\chi) = 0$ or $\nu(\chi) = 1$, and $m_{\mathbb{R}}(\chi) = 2$, if $\nu(\chi) = -1$.*

By Theorem 15.1.1 in [K], we must show that the Schur index $m_{\mathbb{R}}(\chi)$ of the character χ of a representation $\pi \in G^*$ is not equal to 2.

Let π be a quaternionic representation in G^* with character χ. Thus $\nu(\chi) = -1$. By construction (as described in §13 above), χ is induced from a complex-valued (as opposed to real-valued) character ψ of H, where H is the direct product of $(S_n)_\mu$ and C_ℓ^n. In particular, $\mathbb{R}(\psi) = \mathbb{C}$, $\nu(\psi) = 0$, and so, by the Theorem, $m_{\mathbb{R}}(\psi) = 1$. By the Proposition then $m_{\mathbb{R}}(\chi) = 1$. Using the theorem again, we find that $\nu(\chi) = 0$ or $\nu(\chi) = 1$. This is a contradiction. □

Example 3.6. *Let $G = C_3^8 >\!\lhd S_8$. Let*

$$\pi = \theta_{\mu,\rho}$$

denote a representation of G, as constructed above in Theorem 2.1.

- *The real representations π of G are the ones for which*
  - *μ is represented by a character of the form*

$$(1, 1, 1, 1, 1, 1, \omega, \omega^2) \text{ or } (1, 1, ..., 1),$$

    and ρ anything, or

–

$$(1,1,1,1,\omega,\omega,\omega^2,\omega^2), \rho_1 = (\pi_1,\pi_2,\pi_2) \in (S_4)^* \times (S_2)^* \times (S_2)^*,$$

or

–

$$(\omega,\omega,\omega,\omega,\omega^2,\omega^2,\omega^2,\omega^2), \rho_1 = (\pi_2,\pi_2) \in (S_4)^* \times (S_4)^*,$$

or

–

$$(1,1,\omega,\omega,\omega,\omega^2,\omega^2,\omega^2), \rho_1 = (\pi_1,\pi_2,\pi_2) \in (S_2)^* \times (S_3)^* \times (S_3)^*.$$

- *The complex representations of G are: the representations whose characters have at least one complex value. Such representations $\pi = \theta_{\mu,\rho}$ are characterized by the fact that (μ,ρ) is not equivalent to $(\overline{\mu},\rho)$ under the "obvious" S_8-equivalence relation (which can be determined from the equivalence relation for representations in G^*).*

  The complex representations of G are the remaining representations not included in the above list.

- *There are no quaternionic representations of G.*

This was first verified out using a MAPLE program. The claims above follow from Theorem 3.3, the construction in §2.2 above, and the fact (proven in Lemma 2.12) that a representation $\theta_{\rho,\mu}$ is complex if and only if μ is not self-dual. □

3.2. The rationality of representations of $S_n\ wr\ C_\ell$

This section collects, for the pedagogical sake of comparison with general results in the literature, results about the Schur indices of generalized symmetric groups which are particular cases of general results known for any finite group. (A special argument due to M. Benard [B] has shown that the Schur index over $\mathbb{Q}$ of any irreducible representation of G is 1.)

Let $G = S_n\ wr\ C_\ell$. Though we have not yet determined the value of $m_{\mathbb{Q}}(\chi)$ in every case, we have shown that it must be a power of 2. We will eventualy show that $m_{\mathbb{Q}}(\chi) = 1$.

Define the **Euler phi function** $\phi(n)$ to be the number of positive integers less than or equal to n which are relatively prime to n. Though we shall not need it, there is a formula for the value of the phi function: If $n = p_1^{e_1}...p_k^{e_k}$, for distinct primes p_i and integers $e_i \geq 1$, then it is known that

$$\phi(n) = p_1^{e_1-1}...p_k^{e_k-1}(p_1-1)...(p_k-1). \tag{3.1}$$

(For a proof, see any number theory text, for example [HW], §5.5, page 52.)

Define the **exponent** of a finite group G to be the smallest integer $N > 1$ such that $g^N = 1$ for all $g \in G$.

Theorem 3.7. *Let $G = S_n$ wr C_ℓ. Let χ denote an irreducible character of G. Then*

(a) $m_{\mathbb{Q}}(\chi) = 2^k$, for some $k \geq 0$ (depending on χ),

(b) if the exponent of G is divisible by 4 then $m_{\mathbb{Q}(i)}(\chi) = 1$.

Remark 4. *Though they shall not be needed, we can also show that*

1. *$m_{\mathbb{Q}}(\chi)|\phi(\ell)$, where ϕ is the Euler phi function,*
2. *$2m_{\mathbb{Q}}(\chi)|e(G)$, where $e(G)$ is the exponent of G,*
3. *$m_{\mathbb{Q}}(\chi)^2||G|$ (i.e., $m_{\mathbb{Q}}(\chi)^2$ divides $\ell^n n!$),*
4. *if χ is real-valued then $m_{\mathbb{Q}}(\chi) \in \{1, 2\}$.*

In part (2), the exponent of G, $e(G)$, is the exponent of S_n times the exponent of C_ℓ^n, so $e(G) = lcm(1, 2, ..., n) \cdot \ell^n$.

The proofs are included for completeness.

The second and third parts above can probably be strengthened by using a result of Spiegel and Trojan [ST], which gives stronger divisiblity conditions. However, their result is slightly more technical to state.

proof*: To prove (1), we use the following result from field theory (see [L], page 197, ch VIII, §1, Corollary): if $F \subset E \subset \mathbb{C}$ are fields, $\alpha, \beta \in \mathbb{C}$, and if $F(\alpha)/F$ is Galois then $[F(\alpha, \beta) : F(\alpha)]$ divides $[F(\alpha) : F]$.*

Let $\chi = \psi^G$, where ψ is an irreducible character of $(S_n)_\mu \times (C_\ell)^n$ for some $\mu \in (C_\ell^n)^*$. By construction, $\psi = \mathrm{tr}(\sigma \otimes \mu)$, where $\sigma \in ((S_n)_\mu)^*$. Since the character of σ is real-valued (by a result mentioned above), we have $\mathbb{Q}(\psi) = \mathbb{Q}(\mu) \subset \mathbb{Q}(\zeta_\ell)$, where $\zeta_\ell = e^{2\pi i/\ell}$ is a primitive ℓ^{th} root of unity. Using Theorem 10.4 of [I], we find *that $m_{\mathbb{Q}}(\chi)$ divides the degree of $\mathbb{Q}(\chi,\psi)$ over $\mathbb{Q}(\chi)$. By the above mentioned fact from field theory, we find that $m_{\mathbb{Q}}(\chi)$ divides the degree of $\mathbb{Q}(\psi)$ over $\mathbb{Q}$. Since this is a subextension of the cyclotomic extension $\mathbb{Q}(\zeta_\ell)$ over $\mathbb{Q}$ (which, by Galois theory, has degree $\phi(\ell)$, where ϕ is the Euler phi function), we have*

$$[\mathbb{Q}(\psi):\mathbb{Q}] = [\mathbb{Q}(\zeta_\ell):\mathbb{Q}] = \phi(\ell).$$

The claim in part (1) follows.

Parts (2) and (3) follow from the Fein-Yamada Theorem (Theorem 15.1.5 on page 841 of [K]).

Part (4) follows from the Brauer-Speiser theorem (Theorem 15.1.3 on page 840 of [K]).

Before proving the theorem, we state some immediate corollaries,

Corollary 3.8. *Each irreducible character χ of a generalized symmetric group of exponent at least 4 (e.g., of degree at least 3) is realizable over $\mathbb{Q}(i,\chi)$. For such a χ, we have $m_{\mathbb{Q}}(\chi) \in \{1,2\}$.*

proof: The first part is a consequence of part (b) of the above theorem and the definition of the Schur index. The second part follows from the fact that, by the first part and a well-known fact about Schur indices (Corollary 10.2(g) of [I]), $m_{\mathbb{Q}}(\chi)$ must divide the degree $[\mathbb{Q}(i,\chi):\mathbb{Q}(\chi)] = 2$. □

Corollary 3.9. *If $\ell = 2$ then each irreducible character of G is $\mathbb{Q}$-valued. If either ℓ is odd or $\ell/2$ is odd then $m_{\mathbb{Q}}(\chi) \in \{1,2\}$.*

Remark 5. *By Corollary 13.1.2 in §13 of [Se], if each irreducible character of G is $\mathbb{Q}$-valued then each irreducible character of G is $\mathbb{Z}$-valued. For example, each character of $C_2^n >\!\!\lhd S_n$, is $\mathbb{Z}$-valued.*

proof of corollary: Let χ denote an irreducible character of G. The Benard-Schacher Theorem (Theorem 15.1.4 on page 841 of [K]) implies that $\mathbb{Q}(e^{2\pi i/m}) \subset \mathbb{Q}(\chi)$, where $m = m_{\mathbb{Q}}(\chi)$.

Let $\chi = \psi^G$, where ψ is an irreducible character of $(S_n)_\mu \times (C_\ell)^n$ for some $\mu \in (C_\ell^n)^*$. By construction, $\psi = \mathrm{tr}(\sigma \otimes \mu)$, where $\sigma \in ((S_n)_\mu)^*$. Since the character of σ is real-valued (by a result mentioned above), we have $\mathbb{Q}(\psi) = \mathbb{Q}(\mu) \subset \mathbb{Q}(\zeta_\ell)$, where $\zeta_\ell = e^{2\pi i/\ell}$ is a primitive ℓ^{th} root of unity. If $\ell = 2$ then the corollary follows.

It is a general fact about cyclotomic fields that if r, s are relatively prime integers or $(r, s) = 2$ then $\mathbb{Q}(\zeta_r) \cap \mathbb{Q}(\zeta_s) = \mathbb{Q}$. Suppose either ℓ is odd or $\ell/2$ is odd. Since the theorem implies $m = m_{\mathbb{Q}}(\chi)$ is a power of 2, ℓ and m are either relatively prime or $(\ell, m) = 2$, so $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_\ell) = \mathbb{Q}$. The only way that this and $\mathbb{Q}(\zeta_m) \subset \mathbb{Q}(\zeta_\ell)$ are possible is if $m \in \{1, 2\}$. □

proof of theorem: This follow from the Goldschmidt-Isaacs Theorem (Theorem 15.2.1 on page 844 of [K]).

□

Remark 6. *Let χ denote an irreducible character of a finite group G and let $F = \mathbb{Q}(\chi)$.*

- *If χ may be realized over extension fields E and E' then one cannot in general conclude that χ may be realized over the intersection $E \cap E'$.*
- *$\mathbb{Q}(\chi)/\mathbb{Q}$ is a Galois extension with abelian Galois group $Gal(\mathbb{Q}(\chi)/\mathbb{Q}) \subset (\mathbb{Z}/\ell\mathbb{Z})^\times$ (see [I], chapter 10).*

3.2.1. Some special cases

Let $G = S_n\ wr\ C_\ell$. Let $\eta_k(z) = z^k$, for $z \in C_\ell$, $1 \leq k \leq \ell$. Let

$$\mu = (\eta_{e_1}, ..., \eta_{e_n}) \in (C_\ell^n)^*,$$

for some $e_j \in \{0, ..., \ell - 1\}$, and let $\rho \in (S_n)^*_\mu$. Let $x \longmapsto \overline{x}$ denote the mod ℓ map, $\mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z}$.

Lemma 3.10. *Suppose that*
(a) $4|\ell$,
(b) $\overline{e_1 + ... + e_n}$ divides $\overline{\ell/4}$ in $\mathbb{Z}/\ell\mathbb{Z}$.
Then $m_{\mathbb{Q}}(\chi) = 1$, where χ denotes the character of $\theta_{\mu,\rho}$.

Remark 7.

- *In particular, (b) is true if $e_1 + ... + e_n$ is an odd integer.*
- *Suppose $(n, \ell) = 1$ and $4|\ell$. Suppose, on the contrary, $(e_1 + ... + e_n)x \equiv \ell/4 \pmod{\ell}$ is not solvable. In this case, $\overline{e_1 + ... + e_n + n}$ divides $\overline{\ell/4}$ in $\mathbb{Z}/\ell\mathbb{Z}$. This implies, as a corollary of the above result, that the Schur index of a representation closely related to $\theta_{\mu,\rho}$ is equal to 1: $m_{\mathbb{Q}}(\chi) = 1$, where χ denotes the character of $\theta_{\mu\otimes\eta_1,\rho} \equiv \mu \otimes \theta_{\eta_1,\rho}$, where $\mu \otimes \eta_1 = (\mu_1\eta_1, \mu_2\eta_1, ..., \mu_n\eta_1)$.*

proof: By Corollary 3.8, it suffices to prove that $i \in \mathbb{Q}(\chi)$.

Using the Frobenious formula for induced characters (Lemma 2.8), we find that

$$\chi(\vec{v}, 1) = (\dim\ \rho) \sum_{g \in S_n/(S_n)_\mu} \mu^g(\vec{v}) = (\dim\ \rho) \frac{n!}{|(S_n)_\mu)|} \cdot v^{\sum_{j=1}^n e_j},$$

for $\vec{v} = (v, ..., v) \in C_\ell^n$.

Let $d = \ell/4$. By (b), there is a k such that $(e_1 + ... + e_n)k \equiv d \pmod{\ell}$. Taking $v = \zeta_\ell^k$ proves $i \in \mathbb{Q}(\chi)$. □

Example 3.11. *Let $G = S_3\ wr\ C_8$ and $\mu = (\eta_1, \eta_2, \eta_3)$. Since $e_1 + e_2 + e_3 = 6 \equiv -2 \pmod 8$ divides $8/4 = 2$ (mod 8), the lemma implies that the character of $\theta_{\mu,1}$ has Schur index 1 over the rationals. Therefore, $\theta_{\mu,1}$ may be realized over $\mathbb{Q}$.*

Lemma 3.12. *Suppose that $(e_1 + ... + e_n, \ell) = 1$, i.e., they are relatively prime. Then $m_\mathbb{Q}(\chi) = 1$, where χ denotes the character of $\theta_{\mu,\rho}$.*

Remark 8. *Suppose $(n, \ell) = 1$ and $(e_1 + ... + e_n, \ell) > 1$. We have $(e_1 + ... + e_n + n, \ell) = 1$. This implies, as a corollary of the above result, that the Schur index of a representation closely related to $\theta_{\mu,\rho}$ is equal to 1: $m_\mathbb{Q}(\chi) = 1$, where χ denotes the character of $\theta_{\mu\otimes\eta_1,\rho}$, where $\mu \otimes \eta_1 = (\mu_1\eta_1, \mu_2\eta_1, ..., \mu_n\eta_1)$.*

proof: Since $\mathbb{Q}(\chi) \subset \mathbb{Q}(\zeta_\ell)$, it suffices to prove that $\zeta_\ell \in \mathbb{Q}(\chi)$.

As in the above proof, we find that

$$\chi(\vec{v}, 1) = (\dim\ \rho) \frac{n!}{|(S_n)_\mu)|} \cdot v^{\sum_{j=1}^n e_j},$$

for $\vec{v} = (v, ..., v) \in C_\ell^n$.

By hypothesis, there is a k such that $(e_1 + ... + e_n)k \equiv 1 \pmod{\ell}$. Taking $v = \zeta_\ell^k$ proves $\zeta_\ell \in \mathbb{Q}(\chi)$. □

Summarizing the above results, we obtain our main result:

Theorem 3.13. *Let $G = S_n\ wr\ C_\ell$. Let $\mu = (\eta_{e_1}, ..., \eta_{e_n}) \in (C_\ell^n)^*$, for some $e_j \in \{0, ..., \ell - 1\}$, and let $\rho \in (S_n)_\mu^*$. Let χ denotes the character of $\theta_{\mu,\rho}$.*

(a) Suppose that one of the following conditions holds:

- *$4|\ell$ and $\overline{e_1 + ... + e_n}$ divides $\overline{\ell/4}$ in $\mathbb{Z}/\ell\mathbb{Z}$,*

  or

- $(e_1 + ... + e_n, \ell) = 1$,

Then $m_{\mathbb{Q}}(\chi) = 1$.

(b) Suppose that one of the following conditions holds:

- $(n, \ell) = 1$, $4|\ell$, *and* $(e_1 + ... + e_n)x \equiv \ell/4 \pmod{\ell}$ *is not solvable.*

  or

- $(n, \ell) = 1$ *and* $(e_1 + ... + e_n, \ell) > 1$.

Then $m_{\mathbb{Q}}(\chi\eta_1) = 1$.

As mentioned, Benard [B] has shown that $m_{\mathbb{Q}}(\chi) = 1$, for all χ as in the above theorem.

4. Splitting fields

Since the Schur index over $\mathbb{Q}$ of any irreducible character χ of a generalized symmetric group G is equal to 1, each such character is associated to a representation π all of whose matrix coefficients belong to $\mathbb{Q}(\chi)$.

4.1. Which representations can be realized over $\mathbb{Q}$?

Let $\theta_{\mu,\rho} \in G^*$ be the represention defined in §2.1, where $\rho \in ((S_n)_\mu)^*$.

Let $K \subset \mathbb{Q}(\zeta_\ell)$ be a subfield, where ζ_ℓ is a primitive ℓ^{th} root of unity. Assume K contains the field generated by the values of the character of $\theta_{\mu,\rho}$. Assume $K/\mathbb{Q}$ is Galois and let $\Gamma_K = Gal(\mathbb{Q}(\zeta_\ell)/K)$. Note if we regard C_ℓ as a subset of $\mathbb{Q}(\zeta_\ell)$ then there is an induced action of Γ_K on C_ℓ,

$$\sigma : \mu \longmapsto \mu^\sigma, \qquad \mu \in (C_\ell)^*, \quad \sigma \in \Gamma_K,$$

where $\mu^\sigma(z) = \mu(\sigma^{-1}(z))$, $z \in C_\ell$. This action extends to an action on $(C_\ell^n)^* = (C_\ell^*)^n$.

The following result may be called our "key lemma".

Lemma 4.1. *In the notation above,* $\theta_{\mu,\rho} \cong \theta^\sigma_{\mu,\rho}$ *if and only if* μ *is equivalent to* μ^σ *under the action of* S_n *on* $(C_\ell^n)^*$ *described in §2.1.*

proof: This follows immediately from Theorem 2.1. □

Let

$$n_\mu(\chi) = |\{i \mid 1 \le i \le n,\ \mu_i = \chi\}|,$$

where $\mu = (\mu_1, ..., \mu_n) \in (C_\ell^n)^*$ and $\chi \in C_\ell^*$.

Theorem 4.2. *The character of $\theta_{\mu,\rho} \in G^*$ has values in K if and only if $n_\mu(\chi) = n_\mu(\chi^\sigma)$, for all $\sigma \in \Gamma_K$ and all $\chi \in C_\ell^*$.*

proof: Recall

$$\theta_{\mu,\rho} = Ind^G_{(S_n)_\mu \times C_\ell^n}(\rho \otimes \mu).$$

We can extend the action of Γ_K on $(C_\ell^n)^*$ to $((S_n)_\mu \times C_\ell^n)^* = ((S_n)_\mu)^* \times (C_\ell^n)^*$ by making it act trivially on the $(S_n)_\mu)^*$ component. Since G^* is a union of such sets (by the construction in Theorem 2.1), we may extend this action to obtain an action of Γ_K on G^*.

If we know the circumstances under which (the equivalence class of) $\theta_{\mu,\rho}$ is invariant under each $\sigma \in \Gamma_K$ then we will know which $\theta_{\mu,\rho}$ may be realized over $\mathbb{Q}$ since $m_{\mathbb{Q}}(\chi) = 1$, by a result of M. BenardB. To determine these circumstances, we use the "key" Lemma 4.1. Since μ is equivalent to μ^σ if and only if $n_\mu(\chi) = n_\mu(\chi^\sigma)$, for all $\sigma \in \Gamma_K$ and all $\chi \in C_\ell^*$, the theorem follows. □

Example 4.3. *Let $G = S_4\ wr\ C_8$, let $\rho = 1$ and let $\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \in (C_8^3)^*$ be given by*

$$\mu_1(a) = a, \quad \mu_2(a) = a^2, \quad \mu_3(a) = a^3, \quad \mu_4(a) = a^6, \qquad a \in C_8.$$

Let χ denote the character of $\theta_{1,\mu}$. A calculation using a program written in the MAPLE computer algebra package suggests that $\mathbb{Q}(\chi) = \mathbb{Q}(i\sqrt{2})$. (I did not have MAPLE compute all the values of χ, so we will have to assume this is true.) It is interesting to note that -1 is a sum of squares in $\mathbb{Q}(i\sqrt{2})$, so that Fein's improvement over the Goldschmidt-Isaacs Theorem (Theorem 15.2.1 on page 844 of [K]) doesn't imply that $m_{\mathbb{Q}}(\chi) \neq 2$.

In fact, so far, this example does not meet any of the conditions which force $m_{\mathbb{Q}}(\chi) = 1$. In other words, one may ask does $m_{\mathbb{Q}}(\chi) = 2$? The answer, as we shall see, is no. In fact, $m_{\mathbb{Q}}(\chi) = 1$ but to prove this requires some Galois theory.

Let $\zeta_8 = e^{2\pi i/8}$. The Galois group

$$Gal(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times = \{1, -1, 3, -3\},$$

is order $\phi(8) = 4$ *and is given by* $Gal(\mathbb{Q}(\zeta_8)/\mathbb{Q}) = \{1, \tau, \sigma, \sigma\tau\}$*, where* $\tau(a) = \overline{a} = a^{-1}$*,* $\sigma(a) = a^3$*, for* $a \in C_8$*. If you think of* $(\mathbb{Z}/8\mathbb{Z})^\times$ *as a group of integers mod* 8 *then, under the above indicated isomorphism,* τ *corresponds to* -1 *and* σ *to* 3*. Note*

$$Gal(\mathbb{Q}(i\sqrt{2})/\mathbb{Q}) \cong \{1, \tau\}$$

and

$$Gal(\mathbb{Q}(\zeta_8)/\mathbb{Q}(i\sqrt{2})) \cong \{1, \sigma\}.$$

Since $\sigma(\mu_1, \mu_2, \mu_3, \mu_4) = (\mu_3, \mu_6, \mu_1, \mu_2)$*, the above theorem implies that* $\theta_{1,\mu}$ *may be realized over* $\mathbb{Q}(i\sqrt{2})$*. Thus* $m_{\mathbb{Q}}(\chi) = 1$*.*

4.2. Proof of the main result

Let $\Gamma = Gal(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$, so $\Gamma \cong (\mathbb{Z}/\ell\mathbb{Z})^\times$. Let $G = S_n\ wr\ C_\ell$ and identify the set G^* with the set of irreducible complex characters of G. As was explained above, Γ acts on the set G^*. Indeed, let $\gamma \in \Gamma$ be given by $\gamma = \sigma_a$, in the notation of §2.4 above, where $a \in (\mathbb{Z}/\ell\mathbb{Z})^\times$, and let $\mu = (\eta_{e_1}, \eta_{e_2}, ..., \eta_{e_n})$, in the notation of §2.1 above. In this notation, γ sends $\chi = tr(\theta_{\rho,\mu})$ to $\chi^\gamma = tr(\theta_{\rho,\mu^\gamma})$, where

$$\mu^\gamma = (\eta_{ae_1}, \eta_{ae_2}, ..., \eta_{ae_n}).$$

Let $\chi = tr(\theta_{\rho,\mu})$ and let

$$Stab_\Gamma(\chi) = \{\gamma \in \Gamma \mid \chi = \chi^\gamma\} \tag{4.1}$$

denote the stabilizer of χ in Γ. We conclude the following result.

Lemma 4.4. *Let* $\chi = tr(\theta_{\rho,\mu})$*. We have*

$$Stab_\Gamma(\chi) = \{\gamma \in \Gamma \mid (\eta_{ae_1}, \eta_{ae_2}, ..., \eta_{ae_n}) \sim (\eta_{e_1}, \eta_{e_2}, ..., \eta_{e_n})\},$$

where two n-tuples $\vec{v}, \vec{w}$ *satisfy* $\vec{v} \sim \vec{w}$ *if and only if they belong to the same* S_n*-orbit.*

We now "determine" the splitting field of any irreducible character of a generalized symmetric group.

Theorem 4.5. *Let* $\chi = tr(\theta_{\rho,\mu})$ *be an irreducible character of* $G = S_n\ wr\ C_\ell$ *as in §2.1. We have*

$$Gal(\mathbb{Q}(\zeta_\ell)/\mathbb{Q}(\chi)) = Stab_\Gamma(\chi).$$

proof: If $\gamma \in Gal(\mathbb{Q}(\zeta_\ell)/\mathbb{Q}(\chi))$ then $\gamma(x) = x$ for all $x \in \mathbb{Q}(\chi)$. In particular, $\gamma(\chi(g)) = \chi(g)$ for all $g \in G$. Thus, $\gamma \in Stab_\Gamma(\chi)$. Conversely, if $\gamma \in Stab_\Gamma(\chi)$ then γ must fix all elements in $\mathbb{Q}(\chi)$. $\square$

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