A question about tropical Pic(X) as a G-module

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Abstract

This paper addresses the following question: Let X be a tropical curve and let G be a finite subgroup of the automorphism group of X. Let D be a divisor and assume that its equivalence class $[D]$ is G-invariant.
Question: Is there always a $D' \in [D]$ which is G-equivariant?

This was answered by Goldstein, Guralnick, and Joyner [GGJ] in the case of an irreducible algebraic curve over an algebraically closed field. We try to extend the arguments of [GGJ] to the tropical case. For instance, in the tropical case, we prove a tropical analog of Hilbert's Theorem 90 and of Tsen's Theorem (in the abelian case). As our main result, we show that, the answer to the above question is "yes" for the class of tropical "$\mathbb{Q}$-divisors".

1 Introduction

Let X be a connected tropical curve and let G denote a (finite) subgroup of the automorphism group of X.

Let $\mathbb{T} = \mathbb{R} \cup \{\pm\infty\}$.

Let $M(X)$ denote the (tropical) ring of *rational functions* on X, i.e., continuous piecewise-linear $\mathbb{T}$-valued functions with integral slopes. Let $S \subset \mathbb{Q}$ denote any (non-trivial, additive) subgroup and let $M_S(X)$ denote the ring of continuous piecewise-linear functions on X with S-valued slopes. In this case we call S the *slope group*. When $S = \mathbb{Z}$ we omit the subscript. Let $M_S(X)^\sharp$ denote those functions which are not equal to $\pm\infty$ on any part.

Let $\mathrm{Div}(X)$ denote the group of divisors (the free abelian group generated by the points of X) and call $\mathrm{Div}_S(X) = \mathrm{Div}(X) \otimes S$ the *S-divisor classes*. Define the map $\mathrm{div} : M_S(X)^\sharp \to \mathrm{Div}_S(X)$ by $\mathrm{div}(f) = \sum_{P \in X} \mathrm{ord}_P(f) \cdot P$, where $\mathrm{ord}_P(f)$ is the sum of all (S-valued) slopes of f for all edges emanating from P. Let

$$\mathrm{div} : M_S(X)^\sharp \to \mathrm{Div}_S(X),$$

let $\mathrm{Prin}_S(X) = \mathrm{div}(M_S(X)^\sharp)$ denote the subgroup of *principal divisors*, and let $\mathrm{Pic}_S(X) = \mathrm{Div}_S(X)/\mathrm{Prin}_S(X)$ denote the *Picard group* of S-divisor classes. The quotient map $\mathrm{Div}_S(X) \to \mathrm{Pic}_S(X)$ is sometimes denoted $D \mapsto [D]$.

For background on tropical rings, see for example Mikhalkin [M1].

2 Cohomology

The basic idea is to use group cohomology to attack this question. For background on cohomology, we reference Serre [S], ch. VII, or the survey [J].

Let S be a slope group and let $M = M_S(X)^\sharp$. The group G acts on the (tropical-multiplicative) abelian group M via its action on X.

Recall that the 1-*cocycles* on G with coefficients in M are defined by

$$Z^1(G, M) = \{f : G \to M \mid \forall g_1, g_2 \in G,\ f(g_1)g_1 f(g_2) = f(g_1 g_2)\},$$

the 1-*coboundaries* by

$$B^1(G, M) = \{f : G \to M \mid \exists m \in M : \forall g \in G,\ f(g) = gm \cdot m^{-1}\},$$

and the 1-*cohomology* by

$$H^1(G, M) = Z^1(G, M)/B^1(G, M).$$

The following result is roughly analogous to Hilbert's Theorem 90:

Proposition 1 *$H^1(G, M_{\mathbb{Q}}(X)^\sharp) = 0$.*

proof: We prove something slightly more general (while at the same time illustrating some difficulties with computing $H^1(G, M_{\mathbb{Z}}(X)^\sharp)$). Let $\mathbb{Z}_{(|G|)}$ denote the ring of rational numbers $a/|G|^k$, for some $a \in \mathbb{Z}$ and $k \geq 0$ an integer (thisis the localization of $\mathbb{Z}$ at the ideal generated by $|G|$). We shall show that $H^1(G, M_{\mathbb{Z}_{(|G|)}}(X)^\sharp)$ is trivial.

Let $M = M_{\mathbb{Z}_{(|G|)}}(X)^\sharp$, $f \in Z^1(G, M)$ represent a class in $H^1(G, M)$, so $f(g) = f(gx)gf(x)^{-1}$. Taking the product over all $x \in G$, this implies $f(g)^{|G|} = (\prod_{x \in G} f(x)) \cdot g(\prod_{x \in G} f(x))^{-1}$. Taking (tropical) $|G|$-th roots, $f(g) = m \cdot gm^{-1}$, where $m = \prod_{x \in G} f(x)^{1/|G|} \in M_{\mathbb{Z}_{(|G|)}}(X)^\sharp$. This implies $f(g)^{-1} \in B^1(G, M)$. Since $B^1(G, M)$ is an abelian group, $f(g) \in B^1(G, M)$, so the class which f represents is trivial.

The same proof applies if you replace $M = M_{\mathbb{Z}_{(|G|)}}(X)^\sharp$ by $M = M_{\mathbb{Q}}(X)^\sharp$ (since taking $|G|$-th roots at the last step is still allowed). $\square$

Lemma 2 *For any non-trivial subgroup $S \subset \mathbb{Q}$, we have a short exact sequence of $\mathbb{Z}[G]$-modules,*

$$0 \to \mathbb{R} \oplus S^d \to M_S(X)^\sharp \to \mathrm{Prin}(X) \to 0,$$

where d is the number of leaves minus the number of non-terminal vertices adjacent to a leaf.

Remark 1

- *If X is compact then $d = 0$.*
- *This is the tropical analog of the well-known short exact sequences*

$$1 \to F^\times \to F(X)^\times \to \mathrm{Prin}(X) \to 0,$$

  for an irreducible non-singular algebraic curve X over an algebraically closed field F.

proof: The order of a function f at a point is the sum of the outgoing slopes. So for f to be in the kernel, it must have the the sum of the slopes at each point equal to 0.

First, look at the metric graph of X without its leaves. This is a compact set, so f must take a maximum or minimum somewhere. But at the point where the minimum is attained, all outgoing slopes are greater than or equal to 0. Therefore, f must be constant on this part of the graph.

Second, if there multiple leaves at one vertex then the slop of f must sum to 0 at that vertex. Otherwise, the slopes can be arbitrary. □

Recall that the 2-*cocycles* on G with coefficients in M are defined by

$$Z^2(G,M) = \{f : G \times G \to M \mid \forall g_1, g_2, g_3 \in G, \\ g_1 f(g_2, g_3) f(g_1 g_2, g_3)^{-1} f(g_1, g_2 g_3) f(g_1, g_2)^{-1} = 1\},$$

the 2-*coboundaries*[1] by

$$B^2(G,M) = \{f : G \times G \to M \mid \exists h : G \to M : \\ \forall g_1, g_2 \in G,\ f(g_1, g_2) = h(g_1) g_1 h(g_2) h(g_1 g_2)^{-1}\},$$

and the 2-*cohomology* by

$$H^2(G,M) = Z^2(G,M)/B^2(G,M).$$

By definition, we have a short exact sequence

$$0 \to \mathrm{Prin}_S(X) \to \mathrm{Div}_S(X) \to \mathrm{Pic}_S(X) \to 0,$$

as $\mathbb{Z}[G]$-modules. The covariant functor of G-invariants, $M \longmapsto H^0(G,M) = M^G$ is left exact. Therefore, for $M = M_S(X)^\sharp$ we have

$$\begin{aligned} 1 \to H^0(G, \mathbb{R} \oplus S^d) \to H^0(G,M) \to H^0(G, \mathrm{Prin}_S(X)) \\ \to H^1(G, \mathbb{R} \oplus S^d) \to H^1(G,M) \to H^1(G, \mathrm{Prin}_S(X)) \to \\ H^2(G, \mathbb{R} \oplus S^d) \to H^2(G,M) \to H^2(G, \mathrm{Prin}_S(X)) \to \dots , \end{aligned} \tag{1}$$

and

$$\begin{aligned} 0 \to H^0(G, \mathrm{Prin}_S(X)) \to H^0(G, \mathrm{Div}_S(X)) \to H^0(G, \mathrm{Pic}_S(X)) \\ \to H^1(G, \mathrm{Prin}(X)) \to H^1(G, \mathrm{Div}_S(X)) \to H^1(G, \mathrm{Pic}_S(X)) \to \dots . \end{aligned} \tag{2}$$

The following result is a corollary of Proposition 1.

[1] It is straightforward to check that $B^2(G,M) \subset Z^2(G,M)$.

Corollary 3 *For $M = M_{\mathbb{Q}}(X)^\sharp$ we have exact sequences*

$$0 \to (\mathbb{R} \oplus \mathbb{Q}^d)^G \to M^G \to \mathrm{Prin}_{\mathbb{Q}}(X)^G \to H^1(G, \mathbb{R} \oplus \mathbb{Q}^d) \to 0$$

and

$$0 \to H^1(G, \mathrm{Prin}_{\mathbb{Q}}(X)) \to H^2(G, \mathbb{R} \oplus \mathbb{Q}^d) \to \dots .$$

3 Tropical curves

Let X be a connected tropical curve. The following computation will not be needed for our main result.

Proposition 4 $H^1(G, \mathrm{Div}_S(X)) = 0$.

Remark 2

- *The analogous result for algebraic curves is proven in [GGJ] and the proof below follows the same idea.*
- *Much of the proof below (but not all) goes through for $H^2(G, \mathrm{Div}_S(X))$ as well.*

proof: Let GX denote the set of all orbits of points of X under the action of G. Let GX/G denote a complete set of representatives in X of these orbits. For each $P \in X$, let $G_P = \{g \in G \mid gP = P\}$ denote its stabilizer. We have a direct sum decomposition

$$\mathrm{Div}(X) = \oplus_{P \in GX/G} \oplus_{g \in G/G_P} \mathbb{Z}[gP].$$

Let

$$X_{ram} = \{P \in X \mid G_P \neq 1\},$$
$$X_{unram} = \{P \in X \mid G_P = 1\}.$$

Let $\mathrm{Div}_S(X)_{ram}$ and $\mathrm{Div}_S(X)_{unram}$ denote the corresponding $\mathbb{Z}[G]$-submodules of $\mathrm{Div}_S(X)$, so $\mathrm{Div}_S(X) = \mathrm{Div}_S(X)_{unram} \oplus \mathrm{Div}_S(X)_{ram}$. (Unlike for the algebraic curve analog, $\mathrm{Div}_S(X)_{ram}$ can have infinite rank.)

We have

$$\begin{aligned}\mathrm{Div}_S(X)_{unram} &= \oplus_{P \in (GX/G)_{unram}} \oplus_{g \in G} \mathbb{Z}[gP] \\ &= \oplus_{g \in G} \oplus_{P \in (GX/G)_{unram}} \mathbb{Z}[gP],\end{aligned}$$

so $\mathrm{Div}_S(X)_{unram}$ is induced in the sense of Serre [S], page 110. This implies $H^1(G, \mathrm{Div}_S(X)_{unram}) = 0$, by Prop. 1 in §VII.2 of [S].

Recall $H^1(G, L_1 \oplus L_2) = H^1(G, L_1) \oplus H^1(G, L_2)$, for any G-modules L_1, L_2. Suppose that $H^1(G, \mathrm{Div}_S(X)_{ram}) \neq 0$ and let $f \in Z^1(G, \mathrm{Div}_S(X)_{ram})$ denote a non-trivial cocycle. Since G is finite, f is supported on finitely many points. Let $\mathcal{P}$ denote the G-orbit of these points and let L denote the sublattice of $\mathrm{Div}(X)_{ram}$ generated by $\mathcal{P}$. There is no harm in assuming that G is transitive on the basis of L (L is a direct sum of such lattices). So, let $L = \oplus_{g \in G/G_P} \mathbb{Z}[gP]$, for some $P \in GX/G$. So $L \cong ind_H^G(\mathbb{Z})$, where $H \cong G_P$ is the stabilizer of one of the basis elements. Here G acts on the induced module

$$ind_H^G(\mathbb{Z}) = \{f : G \to \mathbb{Z} \mid f(hg) = hf(g), \ \ \forall h \in H, \ g \in G\},$$

which are just the $\mathbb{Z}$-valued functions on $H \backslash G$, by right multiplication. By Shapiro's Lemma (Shatz [Sh], Theorem 8, page 31), $H^1(G, ind_H^G(\mathbb{Z})) \cong H^1(H, \mathbb{Z})$. Now $H^1(H, \mathbb{Z}) = Hom(H, \mathbb{Z}) = 0$, since H is finite.

□

Corollary 5 *The map*

$$H^0(G, \mathrm{Pic}(X)) \to H^1(G, \mathrm{Prin}(X))$$

is surjective.

Proposition 6 *If G is abelian then $H^2(G, M_{\mathbb{Q}}(X)^\sharp)$ is trivial.*

proof: First, assume G is cyclic and let $M = M_{\mathbb{Q}}(X)^\sharp$. Then $H^2(G, M) = M^G/NM$, where $N : M \to M$ is the norm map. Since (tropical) $|G|$-th roots in M always exist, N is surjective, so $H^2(G, M)$ is trivial in this case.

In general, G is a product of cyclic groups. In this case, the result follows by induction and the inflation-restriction sequence $0 \to H^2(G/H, A^H) \to H^2(G, A) \to H^2(H, A)$. □

Remark 3

- *In the case of an algebraic curve, $H^2(G, F^\times(X))) = 1$ by Tsen's theorem (a function field over an algebraically closed field is a C^1 field; see the Corollaries on pages 96 and 109 of Shatz [Sh], or §4 and §7 of chapter X in [S]). An analog of Tsen's theorem for tropical curves is, for G abelian, given in Proposition 6. An analog for general finite groups, if it exists, would be very interesting and useful.*

- *In the case of an algebraic curve, $H^1(G, F(X)^\times) = 1$, by Hilbert's Theorem 90. A tropical curve analog of Hilbert's Theorem 90 is Proposition 1.*

Proposition 7 *$H^2(G, \mathbb{R} \oplus \mathbb{Q}^d)$ is trivial.*

proof: Let $S \subset \mathbb{Q}$ be any $|G|$-divisible subgroup. By Rotman [R] Proposition 10.119, we have $|G| \cdot H^2(G, \mathbb{R} \oplus S^d) = 0$. Since $\mathbb{R} \oplus S^d$ is torsion-free, this means that if $f \in Z^2(G, \mathbb{R} \oplus S^d)$ then $|G| \cdot f \in B^2(G, \mathbb{R} \oplus S^d)$. Since $S \subset \mathbb{Q}$ is $|G|$-divisible, so is $B^2(G, \mathbb{R} \oplus S^d)$. This forces, $f \in B^2(G, \mathbb{R} \oplus S^d)$. $\square$

Remark 4 *If $S \subset \mathbb{Q}$ is finitely generated (which means $S = a\mathbb{Z}$, for some $a \in \mathbb{Q}$) then $H^2(G, \mathbb{R} \oplus S^d)$ is finite. Indeed, $H^2(G, \mathbb{R} \oplus \mathbb{Q}^d) = H^2(G, \mathbb{R}) \oplus H^2(G, S^d)$ and $H^2(G, \mathbb{R}) = 0$ (this is a corollary of the proof above). Since S is finitely generated, so is S^d. Now apply Rotman [R] Corollary 10.120.* $\square$

The following is our main result and implies that the answer to the question raised in the introduction is "yes" for the $\mathbb{Q}$-divisors.

Theorem 8 *If $S = \mathbb{Q}$ then*

$$\mathrm{Div}_{\mathbb{Q}}(X)^G \to \mathrm{Pic}_{\mathbb{Q}}(X)^G$$

is surjective

proof: By Corollary 3, the map $H^1(G, \mathrm{Prin}_{\mathbb{Q}}(X)) \to H^2(G, \mathbb{R} \oplus \mathbb{Q}^d)$ is injective. Therefore, $H^1(G, \mathrm{Prin}_{\mathbb{Q}}(X)) = 0$ by the previous result. The result now follows from (2). $\square$

We return to the case where $S \subset \mathbb{Q}$ can be arbitrary. Consider $\mathrm{Pic}^0_S(X)$, the subgroup of $\mathrm{Pic}_S(X)$ of degree 0 divisors (i.e. the Jacobian). The degree map deg_S defined on $\mathrm{Div}_S(X)$ factors through $\mathrm{Pic}_S(X)$. Let B be $deg(\mathrm{Div}_S(X)^G)$. We identify this with a subgroup of $S \cong \mathrm{Pic}_S(X)/\mathrm{Pic}^0_S(X)$. Clearly, B contains $|G| \cdot S$ and may be bigger.

The analog of Proposition 4 is the following result.

Lemma 9 *$H^1(G, \mathrm{Div}^0_S(X)) \cong S/B$ and $B = bS$, where $b = \min\{|G|/|I| \mid I \subset G$ a stabilizer subgroup$\}$.*

Remark 5 *The analogous result for algebraic curves is also true and the proof below, is basically the same argument as that given in [GGJ].*

proof: First, we prove the isomorphism. Consider the sequence $0 \to \mathrm{Div}_S^0(X) \to \mathrm{Div}_S(X) \to S \to 0$. The map from $\mathrm{Div}_S(X)$ to S is *deg*. Using Proposition 4 yields:

$$(0 \to \mathrm{Div}_S^0(X)^G \to)\ \ \mathrm{Div}_S(X)^G \to S \to H^1(G, \mathrm{Div}_S^0(X)) \to 0,$$

as asserted.

Now we prove the claim about B. Observe that the smallest possible degree of a divisor fixed by G is the size of the smallest orbit of G acting on X. Since each inertia group I is the stabilizer of a point P, the orbit of P under G is in one-to-one correspondence with G/I. Therefore, the size of the smallest possible degree of a divisor fixed by G is equal to the minimum of the $|G|/|I|$, as I ranges over all the inertia subgroups. Since $d(\mathrm{Div}_S(X)^G)$ is an abelian group, this proves the claim. □

Now consider the short exact sequence

$$0 \to \mathrm{Prin}_S(X) \to \mathrm{Div}_S^0(X) \to \mathrm{Pic}_S^0(X) \to 0.$$

Taking fixed points leads to the long exact sequence for cohomology:

$$\begin{array}{c} 0 \to \mathrm{Prin}_S(X)^G \to \mathrm{Div}_S^0(X)^G \xrightarrow{\phi} \mathrm{Pic}_S^0(X)^G \to \\ H^1(G, \mathrm{Prin}_S(X)) \to H^1(G, \mathrm{Div}_S^0(X)) \to H^1(G, \mathrm{Pic}_S^0(X)). \end{array}$$

4 Remarks on the general case

This section, to be written, will explain how to weaken the condition $S = \mathbb{Q}$ to $S = \mathbb{Z}$.

References

[GGJ] D. Goldstein, R. Guralnick, D. Joyner, *A Question about* Pic(X) *as a G-module,* (**Computational Aspects of Algebraic Curves,** (Editor: T. Shaska) Lecture Notes in Computing, WorldScientific, 2005.) Available at:
`http://front.math.ucdavis.edu/math.AG/0407036`

[GK] A. Gathmann, M. Kerber, *A Riemann-Roch theorem in tropical geometry*, Available at:
http://aps.arxiv.org/abs/math/0612129

[J] D. Joyner, *A primer on computational group homology and cohomology*, To appear in the Proceedings of the Gaglione conference, entitled **Aspects of Infinite Groups** (ed. Ben Fine), published by World Scientific Press. This was abridged for publication. A longer version is available at:
http://front.math.ucdavis.edu/0706.0549

[M1] G. Mikhalkin, *Tropical geometry and its applications*, Available at:
http://arxiv.org/abs/math/0601041

[MZ] ——- and I. Zharkov, *Tropical curves, their Jacobians and Theta functions,*, Available at:
http://arxiv.org/pdf/math.AG/0612267v2

[R] J. Rotman, **Advanced modern algebra**, Prentice Hall, 2002.

[S] J.-P. Serre, **Local fields**, Springer-Verlag, 1979.

[Sh] S. Shatz, **Profinite groups, arithmetic, and geometry**, Princeton Univ. Press, 1972.