A question about tropical Pic(X) as a G-module

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Abstract

This paper addresses the following question: Let X be a tropical curve and let G be a finite subgroup of the automorphism group of X. Let D be a divisor and assume that its equivalence class $[D]$ is G-invariant.

Question: Is there always a $D' \in [D]$ which is G-invariant?

This was answered by Goldstein, Guralnick, and Joyner [GGJ] in the case of an irreducible algebraic curve over an algebraically closed field. We try to extend the arguments of [GGJ] to the tropical case. For instance, in the tropical case, we prove a tropical analog of Hilbert's Theorem 90. For our main result, we show that the answer to the above question is "yes" for all tropical curves.

1 Introduction

Let X be a connected tropical curve and let G denote a (finite) subgroup of the automorphism group of X.

Let $\mathbb{T} = \mathbb{R} \cup \{\pm\infty\}$.

Let $M(X)$ denote the (tropical) ring of *rational functions* on X, i.e., continuous piecewise-linear $\mathbb{T}$-valued functions with integral slopes. Let $M(X)^\sharp$ denote those functions which are not equal to $\pm\infty$ on any part.

Let $\mathrm{Div}(X)$ denote the group of divisors (the free abelian group generated by the points of X) and call its elements the *divisor classes* of X. Define the map $\mathrm{div} : M(X)^\sharp \to \mathrm{Div}(X)$ by $\mathrm{div}(f) = \sum_{P\in X} \mathrm{ord}_P(f) \cdot P$, where $\mathrm{ord}_P(f)$ is the sum of all slopes of f for all edges emanating from P. Let

$$\mathrm{div} : M(X)^\sharp \to \mathrm{Div}(X),$$

let $\mathrm{Prin}(X) = \mathrm{div}(M(X)^\sharp)$ denote the subgroup of *principal divisors*, and let $\mathrm{Pic}(X) = \mathrm{Div}(X)/\mathrm{Prin}(X)$ denote the *Picard group* of S-divisor classes. The quotient map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is sometimes denoted $D \mapsto [D]$.

For background on tropical rings, see for example Mikhalkin [M1].

Lemma 1 *("Linear independence of characters") Let $S \subset G$ be any non-empty subset, provided with a fixed ordering. For any $(c_g \mid g \in S) \in M(X)^S$, c_g invertible, there is an $f \in M(X)$ such that*

$$\sum_{g \in S} c_g g(f)$$

is invertible.

We cannot follow the argument of Dedekind (who proved this first in the case of fields, as in Rotman [R], Proposition 4.30), since for the tropical algebra $M(X)$ the cancellation law for addition fails.

proof: This is actully an easy and strightforward consequence of the definitions. $\square$

2 Cohomology

The basic idea is to use group cohomology to attack this question. For background on cohomology, we reference Serre [S], ch. VII, or the survey [J].

Let S be a slope group and let $M = M(X)^\sharp$. The group G acts on the (tropical-multiplicative) abelian group M via its action on X.

Recall that the 1-*cocycles* on G with coefficients in M are defined by

$$Z^1(G,M) = \{f : G \to M \mid \forall g_1, g_2 \in G,\ f(g_1)g_1 f(g_2) = f(g_1 g_2)\},$$

the 1-*coboundaries* by

$$B^1(G,M) = \{f : G \to M \mid \exists m \in M : \forall g \in G,\ f(g) = gm \cdot m^{-1}\},$$

and the 1-*cohomology* by

$$H^1(G,M) = Z^1(G,M)/B^1(G,M).$$

The following result is roughly analogous to Hilbert's Theorem 90.

Proposition 2 *$H^1(G, M(X)^\sharp) = 0$.*

proof: Let $M = M(X)^\sharp$ and pick $f \in Z^1(G,M)$. We assume for this proof that there is a $\phi \in M$ such that $y = \sum_{g\in G} f(g)g(\phi)$ is invertible. (Over a field this is the "linear independence of characters.") We compute

$$h(y) = \sum_{g\in G} hf(g)hg(\phi) = \sum_{g\in G} f(h)^{-1}f(hg)hg(\phi) = f(h)^{-1}y.$$

Therefore a cocycle is a coboundary. $\square$

Here is a proof that $H^1(G, M(X)^\sharp \otimes \mathbb{Q}) = 0$.

Let $M = M(X)^\sharp \otimes \mathbb{Q}$, $f \in Z^1(G,M)$ represent a class in $H^1(G,M)$, so $f(g) = f(gx)gf(x)^{-1}$. Taking the product over all $x \in G$, this implies $f(g)^{|G|} = (\prod_{x\in G} f(x)) \cdot g(\prod_{x\in G} f(x))^{-1}$. Taking (tropical) $|G|$-th roots (since we've tensored by $\mathbb{Q}$ taking $|G|$-th roots is allowed), $f(g) = m \cdot gm^{-1}$, where $m = \prod_{x\in G} f(x)^{1/|G|} \in M_{\mathbb{Z}_{(|G|)}}(X)^\sharp$. This implies $f(g)^{-1} \in B^1(G,M)$. Since $B^1(G,M)$ is an abelian group, $f(g) \in B^1(G,M)$, so the class which f represents is trivial.

Lemma 3 *We have a short exact sequence of $\mathbb{Z}[G]$-modules,*

$$0 \to \mathbb{R} \oplus \mathbb{Z}^d \to M(X)^\sharp \to \mathrm{Prin}(X) \to 0,$$

where d is the number of leaves minus the number of non-terminal vertices adjacent to a leaf.

Remark 1

- *If X is compact then $d = 0$.*
- *This is the tropical analog of the well-known short exact sequences*

$$1 \to F^\times \to F(X)^\times \to \mathrm{Prin}(X) \to 0,$$

for an irreducible non-singular algebraic curve X over an algebraically closed field F.

proof: The order of a function f at a point is the sum of the outgoing slopes. So for f to be in the kernel, it must have the the sum of the slopes at each point equal to 0.

First, look at the metric graph of X without its leaves. This is a compact set, so f must take a maximum or minimum somewhere. But at the point where the minimum is attained, all outgoing slopes are greater than or equal to 0. Therefore, f must be constant on this part of the graph.

Second, if there multiple leaves at one vertex then the slope of f must sum to 0 at that vertex. Otherwise, the slopes can be arbitrary. $\square$

Recall that the 2-*cocycles* on G with coefficients in M are defined by

$$\begin{aligned} Z^2(G,M) = \{f : G\times G \to M \mid \forall g_1,g_2,g_3\in G,& \\ g_1f(g_2,g_3)f(g_1g_2,g_3)^{-1}f(g_1,g_2g_3)f(g_1,g_2)^{-1} = 1\},& \end{aligned}$$

the 2-*coboundaries*[1] by

$$\begin{aligned} B^2(G,M) = \{f : G\times G \to M \mid \exists h : G \to M :& \\ \forall g_1,g_2\in G,\ f(g_1,g_2) = h(g_1)g_1h(g_2)h(g_1g_2)^{-1}\},& \end{aligned}$$

and the 2-*cohomology* by

$$H^2(G,M) = Z^2(G,M)/B^2(G,M).$$

By definition, we have a short exact sequence

$$0 \to \mathrm{Prin}(X) \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0,$$

as $\mathbb{Z}[G]$-modules. The covariant functor of G-invariants, $M \longmapsto H^0(G,M) = M^G$ is left exact. Therefore, for $M = M(X)^\sharp$ we have

$$\begin{aligned} 1 \to H^0(G,\mathbb{R}\oplus\mathbb{Z}^d) \to H^0(G,M) \to H^0(G,\mathrm{Prin}(X))& \\ \to H^1(G,\mathbb{R}\oplus\mathbb{Z}^d) \to H^1(G,M) \to H^1(G,\mathrm{Prin}(X)) \to& \\ H^2(G,\mathbb{R}\oplus S^d) \to H^2(G,M) \to H^2(G,\mathrm{Prin}(X)) \to \ldots\ ,& \end{aligned} \tag{1}$$

and

[1] It is straightforward to check that $B^2(G,M)\subset Z^2(G,M)$.

$$\begin{aligned} 0 \to H^0(G, \mathrm{Prin}(X)) \to H^0(G, \mathrm{Div}(X)) \to H^0(G, \mathrm{Pic}(X)) \\ \to H^1(G, \mathrm{Prin}(X)) \to H^1(G, \mathrm{Div}(X)) \to H^1(G, \mathrm{Pic}(X)) \to ... \, . \end{aligned} \tag{2}$$

The following result is a corollary of Proposition 2.

Corollary 4 *For $M = M(X)^\sharp$ we have exact sequences*

$$0 \to (\mathbb{R} \oplus \mathbb{Z}^d)^G \to M^G \to \mathrm{Prin}(X)^G \to H^1(G, \mathbb{R} \oplus \mathbb{Z}^d) \to 0$$

and

$$0 \to H^1(G, \mathrm{Prin}(X)) \to H^2(G, \mathbb{R} \oplus \mathbb{Z}^d) \to ... \, .$$

3 Tropical curves

Let X be a connected tropical curve. The following computation will not be needed for our main result.

Proposition 5 $H^1(G, \mathrm{Div}(X)) = 0$.

Remark 2
- *The analogous result for algebraic curves is proven in [GGJ] and the proof below follows the same idea.*
- *Much of the proof below (but not all) goes through for $H^2(G, \mathrm{Div}(X))$ as well.*

proof: Let GX denote the set of all orbits of points of X under the action of G. Let GX/G denote a complete set of representatives in X of these orbits. For each $P \in X$, let $G_P = \{g \in G \mid gP = P\}$ denote its stabilizer. We have a direct sum decomposition

$$\mathrm{Div}(X) = \oplus_{P \in GX/G} \oplus_{g \in G/G_P} \mathbb{Z}[gP].$$

Let

$$X_{ram} = \{P \in X \mid G_P \neq 1\},$$
$$X_{unram} = \{P \in X \mid G_P = 1\}.$$

Let $\mathrm{Div}(X)_{ram}$ and $\mathrm{Div}(X)_{unram}$ denote the corresponding $\mathbb{Z}[G]$-submodules of $\mathrm{Div}(X)$, so $\mathrm{Div}(X) = \mathrm{Div}(X)_{unram} \oplus \mathrm{Div}(X)_{ram}$. (Unlike for the algebraic curve analog, $\mathrm{Div}(X)_{ram}$ can have infinite rank.)

We have

$$\begin{aligned}\mathrm{Div}(X)_{unram} &= \oplus_{P\in(GX/G)_{unram}} \oplus_{g\in G} \mathbb{Z}[gP] \\ &= \oplus_{g\in G} \oplus_{P\in(GX/G)_{unram}} \mathbb{Z}[gP],\end{aligned}$$

so $\mathrm{Div}(X)_{unram}$ is induced in the sense of Serre [S], page 110. This implies $H^1(G, \mathrm{Div}(X)_{unram}) = 0$, by Prop. 1 in §VII.2 of [S].

Recall $H^1(G, L_1 \oplus L_2) = H^1(G, L_1) \oplus H^1(G, L_2)$, for any G-modules L_1, L_2. Suppose that $H^1(G, \mathrm{Div}(X)_{ram}) \neq 0$ and let $f \in Z^1(G, \mathrm{Div}(X)_{ram})$ denote a non-trivial cocycle. Since G is finite, f is supported on finitely many points. Let $\mathcal{P}$ denote the G-orbit of these points and let L denote the sublattice of $\mathrm{Div}(X)_{ram}$ generated by $\mathcal{P}$. There is no harm in assuming that G is transitive on the basis of L (L is a direct sum of such lattices). So, let $L = \oplus_{g\in G/G_P}\mathbb{Z}[gP]$, for some $P \in GX/G$. So $L \cong ind_H^G(\mathbb{Z})$, where $H \cong G_P$ is the stabilizer of one of the basis elements. Here G acts on the induced module

$$ind_H^G(\mathbb{Z}) = \{f : G \to \mathbb{Z} \mid f(hg) = hf(g), \ \ \forall h \in H, \ g \in G\},$$

which are just the $\mathbb{Z}$-valued functions on $H\backslash G$, by right multiplication. By Shapiro's Lemma (Shatz [Sh], Theorem 8, page 31), $H^1(G, ind_H^G(\mathbb{Z})) \cong H^1(H, \mathbb{Z})$. Now $H^1(H, \mathbb{Z}) = Hom(H, \mathbb{Z}) = 0$, since H is finite. □

Corollary 6 *The map*

$$\mathrm{Pic}(X)^G = H^0(G, \mathrm{Pic}(X)) \to H^1(G, \mathrm{Prin}(X))$$

is surjective.

Remark 3 *Claim: If G is abelian then $H^2(G, M(X)^\sharp \otimes \mathbb{Q})$ is trivial.*

proof of claim: First, assume G is cyclic and let $M = M(X)^\sharp \otimes \mathbb{Q}$. Then $H^2(G, M) = M^G/NM$, where $N : M \to M$ is the norm map. Since (tropical) $|G|$-th roots in M always exist, N is surjective, so $H^2(G, M)$ is trivial in this case.

In general, G is a product of cyclic groups. In this case, the result follows by induction and the inflation-restriction sequence $0 \to H^2(G/H, A^H) \to H^2(G, A) \to H^2(H, A)$. □

Remark 4 • *In the case of an algebraic curve, $H^2(G, F^\times(X))) = 1$ by Tsen's theorem (a function field over an algebraically closed field is a C^1 field; see the Corollaries on pages 96 and 109 of Shatz [Sh], or §4 and §7 of chapter X in [S]). An analog of Tsen's theorem for tropical curves would be the computation of $H^2(G, M(X)^\sharp)$. Such an analog, if it exists, would be very interesting and useful.*

- *In the case of an algebraic curve, $H^1(G, F(X)^\times) = 1$, by Hilbert's Theorem 90. A tropical curve analog of Hilbert's Theorem 90 is Proposition 2.*

Proposition 7 *$H^2(G, \mathbb{R} \oplus \mathbb{Z}^d) \cong H^2(G, \mathbb{Z})^r$, where r is the number of reduced orbits of G on the set of leaves of X.*

In particular, if X is compact then $H^2(G, \mathbb{R} \oplus \mathbb{Z}^d) = 0$.

proof: By Rotman [R] Proposition 10.119, we have $|G| \cdot H^2(G, \mathbb{R} \oplus \mathbb{Z}^d) = 0$. Since $\mathbb{R} \oplus \mathbb{Z}^d$ is torsion-free, this means that if $f \in Z^2(G, \mathbb{R} \oplus \mathbb{Z}^d)$ then $|G| \cdot f \in B^2(G, \mathbb{R} \oplus \mathbb{Z}^d)$. Since $\mathbb{Z}^d$ is, as a G-module, the direct sum of r induced modules, each of the form $ind_H^G(\mathbb{Z})$, by Shapiro's lemma, $H^2(G, \mathbb{Z}^d) \cong H^2(G, \mathbb{Z})^r$. □

Corollary 8 *$H^1(G, \mathrm{Prin}(X)) = 0$.*

proof: The tropical curve decomposes into a disjoint union of a compact tropical curve X_c and it's leaves X_{nc}. The G-action respects this decomposition, and moreover, as G-modules, we have

$$\mathrm{Prin}(X) = \mathrm{Prin}(X_c) \oplus \mathrm{Prin}(X_{nc}).$$

Since on the leaves, all divisors are principal, we have (as a corollary of the proof of Proposition 5), $H^1(G, \mathrm{Prin}(X_{nc})) = 0$. On the other hand, in the compact case, we have $H^1(G, \mathrm{Prin}(X_c)) = 0$, due to previous Proposition 7 and Corollary 4. Since $H^1(G, \mathrm{Prin}(X)) = H^1(G, \mathrm{Prin}(X_c)) \oplus H^1(G, \mathrm{Prin}(X_{nc}))$, the result follows. □

The following is our main result and implies that the answer to the question raised in the introduction is "yes" for all tropical curves.

Theorem 9 *If X is compact then*

$$\mathrm{Div}_{\mathbb{Q}}(X)^G \to \mathrm{Pic}_{\mathbb{Q}}(X)^G$$

is surjective

proof: By the above corollary, $H^1(G, \mathrm{Prin}(X)) = 0$. The result now follows from (2). $\Box$

Consider $\mathrm{Pic}^0(X)$, the subgroup of $\mathrm{Pic}(X)$ of degree 0 divisors (i.e. the Jacobian). The degree map *deg* defined on $\mathrm{Div}(X)$ factors through $\mathrm{Pic}(X)$. Let B be $deg(\mathrm{Div}(X)^G)$. We identify this with a subgroup of $S \cong \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. Clearly, B contains $|G| \cdot \mathbb{Z}$ and may be bigger.

The analog of Proposition 5 is the following result.

Lemma 10 *$H^1(G, \mathrm{Div}^0(X)) \cong \mathbb{Z}/B$ and $B = b\mathbb{Z}$, where $b = \min\{|G|/|I| \mid I \subset G$ a stabilizer subgroup$\}$.*

Remark 5 *The analogous result for algebraic curves is also true and the proof below, is basically the same argument as that given in [GGJ].*

proof: First, we prove the isomorphism. Consider the sequence $0 \to \mathrm{Div}^0(X) \to \mathrm{Div}(X) \to \mathbb{Z} \to 0$. The map from $\mathrm{Div}(X)$ to $\mathbb{Z}$ is *deg*. Using Proposition 5 yields:

$$(0 \to \mathrm{Div}^0(X)^G \to)\ \ \mathrm{Div}(X)^G \to \mathbb{Z} \to H^1(G, \mathrm{Div}^0(X)) \to 0,$$

as asserted.

Now we prove the claim about B. Observe that the smallest possible degree of a divisor fixed by G is the size of the smallest orbit of G acting on X. Since each inertia group I is the stabilizer of a point P, the orbit of P under G is in one-to-one correspondence with G/I. Therefore, the size of the smallest possible degree of a divisor fixed by G is equal to the minimum of the $|G|/|I|$, as I ranges over all the inertia subgroups. Since $d(\mathrm{Div}(X)^G)$ is an abelian group, this proves the claim. $\Box$

Now consider the short exact sequence

$$0 \to \mathrm{Prin}(X) \to \mathrm{Div}^0(X) \to \mathrm{Pic}^0(X) \to 0.$$

Taking fixed points leads to the long exact sequence for cohomology:

$$\begin{array}{r} 0 \to \mathrm{Prin}(X)^G \to \mathrm{Div}^0(X)^G \xrightarrow{\phi} \mathrm{Pic}^0(X)^G \to \\ H^1(G, \mathrm{Prin}(X)) \to H^1(G, \mathrm{Div}^0(X)) \to H^1(G, \mathrm{Pic}^0(X)). \end{array}$$

References

[GGJ] D. Goldstein, R. Guralnick, D. Joyner, *A question about* Pic(X) *as a G-module,* (**Computational Aspects of Algebraic Curves,** (Editor: T. Shaska) Lecture Notes in Computing, WorldScientific, 2005.) Available at:
http://front.math.ucdavis.edu/math.AG/0407036

[GK] A. Gathmann, M. Kerber, *A Riemann-Roch theorem in tropical geometry*, Available at:
http://aps.arxiv.org/abs/math/0612129

[J] D. Joyner, *A primer on computational group homology and cohomology*, To appear in the Proceedings of the Gaglione conference, entitled **Aspects of Infinite Groups** (ed. Ben Fine), published by World Scientific Press. This was abridged for publication. A longer version is available at:
http://front.math.ucdavis.edu/0706.0549

[M1] G. Mikhalkin, *Tropical geometry and its applications*, Available at:
http://arxiv.org/abs/math/0601041

[MZ] ——- and I. Zharkov, *Tropical curves, their Jacobians and theta functions,*, Available at:
http://arxiv.org/pdf/math.AG/0612267v2

[R] J. Rotman, **Advanced modern algebra**, Prentice Hall, 2002.

[S] J.-P. Serre, **Local fields**, Springer-Verlag, 1979.

[Sh] S. Shatz, **Profinite groups, arithmetic, and geometry**, Princeton Univ. Press, 1972.